SUGGESTED SOLUTIONS TO CATEGORY THEORY EXERCISES

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ABSTRACT. This document provides suggested solutions to the exercises from [Rie16]. These solutions are not guaranteed to be correct and are to be used at the reader's own discretion. We strongly encourage readers to attempt the problems independently and to seek alternative proofs or counterexamples.

Exercise (1.2.ii)**.** (1) Show that a morphism $f : x \to y$ is a split epimorphism in a category C if and only if for all $c \in \mathsf{C}$, the post-composition function $f_* : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$ is surjective.

(2) Argue by duality that f is a split monomorphism if and only if for all $c \in \mathsf{C}$, the pre-composition function $f_* : C(y, c) \to C(x, c)$ is surjective.

Proof. (1) $\Longrightarrow$**:** Assume that $f : x \to y$ is a split epimorphism. Then there exists a morphism $g : y \to x$ such that $fg = \mathrm{id}_y$. Thus for any morphism $h : c \to y$, we have

$$f_*(gh) = f(gh) = (fg)h = \mathrm{id}_y h = h,$$

which means that $f_* : \mathsf{C}(c, x) \to \mathsf{C}(c, y)$ is surjective.

$\Longleftarrow$**:** Take $c = y$. Since $f_* : \mathsf{C}(y, x) \to \mathsf{C}(y, y)$ is surjective, there exists a morphism $g : y \to x$ such that $f_*(g) = \mathrm{id}_y$, i.e., $fg = \mathrm{id}_y$. It means that f is a split epimorphism.

(2) By definition, $f : x \to y$ is a split monomorphism in C if and only if $f^{\mathrm{op}} : y \to x$ is a split epimorphism in C^{op}. By (1), f^{op} is a split epimorphism in C^{op} if and only if $f^{\mathrm{op}}_* : \mathsf{C}^{\mathrm{op}}(c, y) \to \mathsf{C}(c, x)$ is surjective. The latter assertion is equivalent to that $f_* : \mathsf{C}(y, c) \to \mathsf{C}(x, c)$ is surjective. We are done. □

Exercise (1.2.iv)**.** What are the monomorphisms in the category of fields?

Solution. Since every morphism in the category of fields is injective, and since every injective homomorphism between fields is a monomorphism, it follows that all morphisms in this category are monomorphisms. □

Remark. In the above exercise, we use the fact that every injective map between fields is a monomorphism. In fact, every injective homomorphism in the category of rings is a monomorphism. This can be proved by mimicking the argument in Exercise (1.2.v)(1) immediately below.

Exercise (1.2.v)**.** Show that the inclusion $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$ is both a monomorphism and an epimorphism in the category Ring of rings. Conclude that a map that is both monic and epic need not be an isomorphism.

Proof. (1) We prove that the inclusion $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$ is a monomorphism. In fact, if $f, g : A \rightrightarrows \mathbb{Z}$ are two homomorphisms of rings such that $if = ig$, then for all $a \in A$, $f(a) = g(a)$ in $\mathbb{Q}$ and thus $f(a) = g(a)$ in $\mathbb{Z}$. Thus $f = g$, which means that i is a monomorphism.

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(2) We show that the inclusion $i : \mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism. Suppose that $f, g : \mathbb{Q} \rightrightarrows A$ are two homomorphisms of rings such that $fi = gi$, i.e., $f|_{\mathbb{Z}} = g|_{\mathbb{Z}}$. For any integer q, we have $f(q)f(1/q) = 1 = f(1/q)f(q)$. Thus $f(q)$ is invertible with inverse $f(1/q)$. Similar for g. Since $f(q) = g(q)$, we obtain $f(1/q) = g(1/q)$. Thus for any integers p and q, we have

$$f\left(\frac{p}{q}\right) = f(p)f\left(\frac{1}{q}\right) = g(p)g\left(\frac{1}{q}\right) = g\left(\frac{p}{q}\right).$$

This means that $f = g$ and that i is an epimorphism. □

Exercise (1.3.i)**.** What is a functor between groups, regarded as one-object categories?

Solution. A functor is the same as a homomorphism between groups. Let G and H be two groups, and let $F : BG \to BH$ be a functor. Because BG and BH each have a single object, the object-mapping condition for a functor is vacuous. The morphism-mapping condition for F is that F is a map from $G = \mathrm{End}_{BG}(*_G)$ to $H = \mathrm{End}_{BH}(*_H)$ satisfying that $F(gg') = F(g)F(g')$ and $F(e_G) = e_H$, which means that $F : G \to H$ is a homomorphism between groups. □

Exercise (1.3.ix)**.** For any group G, we may define other groups:

- the center $Z(G) = \{h \in G \mid hg = gh \ \forall \ g \in G\}$, a subgroup of G,
- the commutator subgroup $C(G)$, the subgroup of G generated by elements $ghg^{-1}h^{-1}$ for any $g, h \in G$, and
- the automorphism group $\mathrm{Aut}(G)$, the group of isomorphisms $\phi : G \to G$ in Group.

Trivially, all three constructions define a functor from the discrete category of groups (with only identity morphisms) to Group. Are these constructions functorial in

- the isomorphisms of groups? That is, do they extend to functors $\mathsf{Group}_{\mathrm{iso}} \to \mathsf{Group}$?
- the epimorphisms of groups [1]? That is, do they extend to functors $\mathsf{Group}_{\mathrm{epi}} \to \mathsf{Group}$?
- all homomorphisms of groups? That is, do they extend to functors $\mathsf{Group} \to \mathsf{Group}$?

Solution. Answers are given in the following table:

	$\mathsf{Group}_{\mathrm{iso}} \to \mathsf{Group}$	$\mathsf{Group}_{\mathrm{epi}} \to \mathsf{Group}$	$\mathsf{Group} \to \mathsf{Group}$
$Z(G)$	Yes	Yes	No
$C(G)$	Yes	Yes	Yes
$\mathrm{Aut}(G)$	Yes	Yes	No

(1) We consider the first two cases of $Z(G)$. In $\mathsf{Group}_{\mathrm{iso}}$ and $\mathsf{Group}_{\mathrm{epi}}$, every morphism $f : G_1 \to G_2$ is surjective. Thus for any $g_2 \in G_2$, there exists an element $g_1 \in G_1$ such that $f(g_1) = g_2$. Then for any $h \in G_1$,

$$f(h)g_2 = f(h)f(g_1) = f(hg_1) = f(g_1h) = f(g_1)f(h) = g_2f(h).$$

Thus $f(h) \in Z(G_2)$ for any $h \in G_1$. So a morphism $f : G_1 \to G_2$ induces a morphism $f : Z(G_1) \to Z(G_2)$. Then we can check easily that the construction $G \mapsto Z(G)$ extends to functors $\mathsf{Group}_{\mathrm{iso}} \to \mathsf{Group}$ and $\mathsf{Group}_{\mathrm{epi}} \to \mathsf{Group}$.

[1] A non-trivial theorem demonstrates that a homomorphism $\phi : G \to H$ is an epimorphism in Group if and only if its underlying function is surjective. **Waring**: But it is false in Ring. For example, $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism in Ring but is not surjective.

(2) We consider the construction $C(G)$. Clearly,
$$f(ghg^{-1}h^{-1}) = f(g)f(h)f(g)^{-1}f(h)^{-1}.$$
Thus $f : G_1 \to G_2$ induces a morphism $f : C(G_1) \to C(G_2)$. Then we can check easily that the construction $G \mapsto C(G)$ extends to functors in all these three cases.

(3) We consider the entry ($\mathrm{Aut}(G)$, $\mathsf{Group}_{\mathrm{iso}} \to \mathsf{Group}$). Note that an isomorphism $f : G_1 \to G_2$ induces a homomorphism
$$\begin{aligned} a_f : \mathrm{Aut}(G_1) &\longrightarrow \mathrm{Aut}(G_2) \\ \sigma &\longmapsto f\sigma f^{-1}. \end{aligned}$$
We can check easily that
$$\begin{aligned} \mathsf{Group}_{\mathrm{iso}} &\longrightarrow \mathsf{Group} \\ G &\longmapsto \mathrm{Aut}(G) \\ (f : G_1 \mapsto G_2) &\longmapsto (a_f : \mathrm{Aut}(G_1) \to \mathrm{Aut}(G_2)) \end{aligned}$$
is a functor.

(4) We consider the entry ($\mathrm{Aut}(G)$, $\mathsf{Group}_{\mathrm{epi}} \to \mathsf{Group}$). As we have seen above, an isomorphism $f : G_1 \to G_2$ induces a homomorphism
$$\begin{aligned} a_f : \mathrm{Aut}(G_1) &\longrightarrow \mathrm{Aut}(G_2) \\ \sigma &\longmapsto f\sigma f^{-1}. \end{aligned}$$
If f is not an isomorphism, then we define $a_f : \mathrm{Aut}(G_1) \to \mathrm{Aut}(G_2)$ to be the trivial map, i.e.,
$$a_f(\sigma) = \mathrm{id}_{G_2}, \ \forall\, \sigma \in \mathrm{Aut}(G_1).$$
We can then check easily that
$$\begin{aligned} \mathsf{Group}_{\mathrm{epi}} &\longrightarrow \mathsf{Group} \\ G &\longmapsto \mathrm{Aut}(G) \\ (f : G_1 \mapsto G_2) &\longmapsto (a_f : \mathrm{Aut}(G_1) \to \mathrm{Aut}(G_2)) \end{aligned}$$
is a functor by noting that a_{gf} is trivial if and only if either a_f or a_g is trivial.

(5) Now, we give a counterexample to the two "No" entries. Let $k = \mathbb{F}_{11}$ be the finite field with 11 elements. We consider the following groups
$$G = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \middle| x \in k^\times, y \in k \right\},\ N = \left\{ \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \middle| y \in k \right\},\ H = \left\{ \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \middle| x \in k^\times \right\}.$$
We have two homomorphisms: the inclusion $i : H \to G$ and the projection (check that it is a homomorphism)[2]
$$\begin{aligned} p : G &\longrightarrow H \\ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} &\longmapsto \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$
We claim that $Z(H) = H$, $Z(G) = \{1\}$, $\mathrm{Aut}(H) \simeq \mathbb{Z}/4\mathbb{Z}$, and $\mathrm{Aut}(G) \simeq G$, whose proof will be given in the **Remark** below.

[2] Here, N is a normal subgroup of G and G/N is isomorphic to H. The group G is the so-called *semi-direct product* of N and H, written as $G = N \rtimes H$ or $G = H \ltimes N$. For more information on semi-direct products, see Wiki.

Assume that $G \mapsto Z(G)$ defines a functor $Z : \text{Group} \to \text{Group}$. Then $Z(p)Z(i) = Z(pi) = Z(\mathrm{id}_H) = \mathrm{id}_{Z(H)}$. Thus $Z(i) : Z(H) \to Z(G)$ is an injective homomorphism, a contradiction. Hence $G \mapsto Z(G)$ doest not extend to a functor $Z : \text{Group} \to \text{Group}$.

Assume that $G \mapsto \mathrm{Aut}(G)$ defines a functor $a : \text{Group} \to \text{Group}$. Then $a(p)a(i) = a(pi) = a(\mathrm{id}_H) = \mathrm{id}_{\mathrm{Aut}(H)}$. Thus $a(i) : a(H) \to a(G)$ is an injective homomorphism. But $\#\mathrm{Aut}(H) = 4$, $\#\mathrm{Aut}(G) = \#G = 110$, $4 \nmid 110$. So $\mathrm{Aut}(H)$ is not a subgroup of $\mathrm{Aut}(G)$, a contradiction. Hence $G \mapsto \mathrm{Aut}(G)$ doest not extend to a functor $\text{Group} \to \text{Group}$. □

Remark. In this remark, we prove the claim in the above proof (In fact, the same proof works for any finite field k).

Suppose that $\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}$ is an element in $Z(G)$. Then it commutes with $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, which means $\begin{pmatrix} -x & -y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -x & y \\ 0 & 1 \end{pmatrix}$. So $y = 0$. Since $\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}$ commutes with $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, we obtain $\begin{pmatrix} x & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x & x \\ 0 & 1 \end{pmatrix}$. So $x = 1$. Hence $Z(G) = \{1\}$.

Note that H is isomorphic to $\mathbb{F}_{11}^\times \simeq \mathbb{Z}/10\mathbb{Z}$, a cyclic group. Hence $Z(H) = H$ and $\mathrm{Aut}(H) \simeq \mathrm{Aut}(\mathbb{Z}/10\mathbb{Z}) \simeq (\mathbb{Z}/10\mathbb{Z})^\times \simeq \mathbb{Z}/4\mathbb{Z}$.

Now, we show that the following map is an isomorphism

$$\begin{aligned} I : G &\longrightarrow \mathrm{Aut}(G) \\ h &\longmapsto (I_g : h \mapsto ghg^{-1}). \end{aligned}$$

In fact, for any group G, we have such a homomorphism whose kernel is $Z(G)$ (check it by yourself if you don't know it or have forgotten the proof). In our case, $Z(G) = \{1\}$ and thus I is injective.

It remains to show that I is surjective. In other words, we have to show that every $\phi \in \mathrm{Aut}(G)$ is the conjugation given by some $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$. By Sylow's theorem, N is the unique Sylow 11-subgroup of G, which implies $\phi(N) = N$. Thus

$$\phi \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$$

for some $a \in k^\times$. It follows that for any $y, c \in k$,

$$\phi \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & ay \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & c \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & c \\ 0 & 1 \end{pmatrix}^{-1}. \tag{1}$$

Fix a generator x_0 of the cyclic group $k^\times$. Write

$$\phi \begin{pmatrix} x_0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x' & y' \\ 0 & 1 \end{pmatrix}.$$

Set

$$b := y'(1 - x_0)^{-1}.$$

We claim that

$$\phi \begin{pmatrix} x_0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1}. \tag{2}$$

It suffices to show $x' = x_0$. From the matrix identity

$$\begin{pmatrix} x_0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x_0 & x_0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x_0 & 0 \\ 0 & 1 \end{pmatrix},$$

we see that the images of these products under the homomorphism ϕ must coincide. Therefore,

$$\begin{pmatrix} x' & y' \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & ax_0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x' & y' \\ 0 & 1 \end{pmatrix}.$$

Computing both sides yields:

$$\begin{pmatrix} x' & ax' + y' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x' & ax_0 + y' \\ 0 & 1 \end{pmatrix},$$

Equating the entries gives $x' = x_0$, which completes the proof of (2). Since x_0 is a generator of the cyclic group $k^\times$, we have

$$\phi\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1}, \text{ for any } x \in k^\times. \tag{3}$$

Noting that

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix},$$

we obtain from (1) and (3) that

$$\begin{aligned} \phi\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1}. \end{aligned}$$

We are done.

Exercise (1.4.i)**.** Suppose $\alpha : F \Rightarrow G$ is a natural isomorphism. Show that the inverses of the component morphisms define the components of a natural isomorphism $\alpha^{-1} : G \Rightarrow F$.

Proof. By definition, for any $c \in \mathsf{C}$, every component α_c is an isomorphism. Since α is natural, we have $Gf \circ \alpha_c = \alpha_{c'} \circ Ff$. Thus we have $\alpha_{c'}^{-1} \circ Gf = Ff \circ \alpha_c^{-1}$.

$$\begin{array}{ccccc} Fc & \xrightarrow{\alpha_c} & Gc & \xrightarrow{\alpha_c^{-1}} & Fc \\ {\scriptstyle Ff}\downarrow & & {\scriptstyle Gf}\downarrow & & \downarrow{\scriptstyle Ff} \\ Fc' & \xrightarrow[\alpha_{c'}]{} & Gc' & \xrightarrow[\alpha_{c'}^{-1}]{} & Fc'. \end{array}$$

It means that $\alpha^{-1} : G \Rightarrow F$ is a natural transformation in which every component α_c^{-1} is an isomorphism. Thus α^{-1} is a natural isomorphism. □

Exercise (1.4.ii)**.** What is a natural transformation between a parallel pair of functors between groups, regarded as one-object categories?

Solution. Let G and H be two groups regarded as one-object categories. Recall that functors from BG to BH are in fact the same as group homomorphisms. Let $\phi, \psi : G \rightrightarrows H$ be two homomorphisms, regarded as functors.

$$\begin{array}{ccc} * & & \\ \Big\downarrow g & & \\ * & & \end{array} \qquad \begin{array}{ccc} * & \xrightarrow{h} & * \\ \phi(g)\Big\downarrow & & \Big\downarrow \psi(g) \\ * & \xrightarrow{h} & * \end{array}$$

By definition, a natural transformation $h : \phi \Rightarrow \psi$ is an endomorphism h of the unique object of BH (i.e., an element h in the group H) such that for any morphism g in the category BG (i.e., an element g in the group G),

$$\psi(g)h = h\phi(g).$$

□

Exercise (1.5.xi)**.** Consider the functors $\mathsf{Ab} \to \mathsf{Group}$ (inclusion), $\mathsf{Ring} \to \mathsf{Ab}$ (forgetting the multiplication), $(-)^\times : \mathsf{Ring} \to \mathsf{Group}$ (taking the group of units), $\mathsf{Ring} \to \mathsf{Rng}$ (inclusion), $\mathsf{Field} \to \mathsf{Ring}$ (inclusion), and $\mathsf{Mod}_R \to \mathsf{Ab}$ (forgetful). Determine which functors are full, which are faithful, and which are essentially surjective. Do any define an equivalence of categories? (Warning: A few of these questions conceal research-level problems, but they can be fun to think about even if full solutions are hard to come by.)

Solution. Answers are given in the following table:

	full	faithful	essentially surjective	equivalence
$\mathsf{Ab} \to \mathsf{Group}$	Yes	Yes	No	No
$\mathsf{Ring} \to \mathsf{Ab}$	No	Yes	No	No
$(-)^\times : \mathsf{Ring} \to \mathsf{Group}$	No	No	No	No
$\mathsf{Ring} \to \mathsf{Rng}$	No	Yes	No	No
$\mathsf{Field} \to \mathsf{Ring}$	Yes	Yes	No	No
$\mathsf{Mod}_R \to \mathsf{Ab}$	$\mathbb{Z} \to R$ is epic	Yes	$R \to \mathbb{Z}$ is surjective	$\mathbb{Z}$

The properties of the forgetful functor $\mathsf{Mod}_R \to \mathsf{Ab}$ vary with the ring R and will be characterized below. Each "Yes" has a trivial justification (ensure you understand why it is trivial). Counterexamples are provided for all "No" answers (These functors all fail to be equivalences of categories due to the "No" entries present in the first three columns):

(1) The inclusion $\mathsf{Ab} \to \mathsf{Group}$ is not essentially surjective. A counterexample: $\mathrm{GL}_2(\mathbb{R})$.
(2) The functor $\mathsf{Ring} \to \mathsf{Ab}$ (forgetting the multiplication) is not full. A counterexample: $\mathrm{Hom}_{\mathsf{Ring}}(\mathbb{Z}, \mathbb{Z}) = \{\mathrm{id}\}$, while $\mathrm{Hom}_{\mathsf{Ab}}(\mathbb{Z}, \mathbb{Z}) \simeq \mathbb{Z}$.
(3) The functor $\mathsf{Ring} \to \mathsf{Ab}$ (forgetting the multiplication) is not essentially surjective. A counterexample: There is no ring structure on the abelian group $\mathbb{Q}/\mathbb{Z}$. Otherwise, let $u := p/q$ be the multiplicative identity in $\mathbb{Q}/\mathbb{Z}$ with $p, q \in \mathbb{Z}$. Then the sum[3] of q-copies of u is p, which is zero in $\mathbb{Q}/\mathbb{Z}$. It follows that the sum of q-copies of any element x is zero in $\mathbb{Q}/\mathbb{Z}$:

$$x + \cdots + x = ux + \cdots + ux = (u + \cdots + u)x = 0.$$

[3] Here, one should use the addition in $\mathbb{Q}/\mathbb{Z}$ rather than the multiplication in $\mathbb{Q}/\mathbb{Z}$, as the latter may not be inherited from $\mathbb{Q}$.

In other words, the sum of q-copies of any rational number is an integer, which is not the case (for example, take $x = 1/(1+q)$).

(4) The functor $(-)^\times$: Ring → Group is not full. A counterexample: $\mathrm{Hom}_{\mathsf{Ring}}(\mathbb{Z},\mathbb{Z}) = \{\mathrm{id}\}$, while $\mathrm{Hom}_{\mathsf{Group}}(\mathbb{Z}^\times,\mathbb{Z}^\times) \simeq \mathrm{Hom}_{\mathsf{Group}}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$.

(5) The functor $(-)^\times$: Ring → Group is not faithful. A counterexample: Consider two ring homomorphisms from $\mathbb{Q}[X]$ to $\mathbb{Q}$ that restrict to the identity on $\mathbb{Q}$, defined by $X \mapsto 0$ and $X \mapsto 1$, respectively.

(6) The functor $(-)^\times$: Ring → Group is not essentially surjective. A counterexample: $\mathbb{Z}/5\mathbb{Z}$ is not isomorphic to the group of units in a ring. Otherwise, suppose that R is a ring such that $R^\times = \{1, g, g^2, g^3, g^4\}$ is a cyclic group of order 5. Since $R^\times$ does not contain $\{1,-1\}$ as a subgroup, the ring R is of characteristic 2. Consider the surjective homomorphism of rings

$$\begin{aligned} \phi : \mathbb{F}_1[X] &\longrightarrow \mathbb{F}_2[g] \\ f(X) &\longmapsto f(g). \end{aligned}$$

Since $g^5 = 1$, the kernel of ϕ contains (X^5-1). So $\ker(\phi)$ is either (X^5-1), or the ideals generated by the irreducible factors of X^5-1, i.e., $(X-1)$ or $(X^4+X^3+X^2+X+1)$ (check that it is irreducible). But $\ker(\phi) \neq (X-1)$ as $g \neq 1$. Thus

$$\mathbb{F}_2[g] \simeq \mathbb{F}_2[X]/(X^4+X^3+X^2+X+1) \simeq \mathbb{F}_{16},$$

or

$$\begin{aligned} \mathbb{F}_2[g] &\simeq \mathbb{F}_2[X]/(X^5-1) \\ &\simeq \mathbb{F}_2[X]/((X-1)(X^4+X^3+X^2+X+1)) \\ &\simeq \mathbb{F}_2[X]/(X-1) \times \mathbb{F}_2[X]/(X^4+X^3+X^2+X+1) \\ &\simeq \mathbb{F}_2 \times \mathbb{F}_{16}, \end{aligned}$$

where the third isomorphism holds by the Chinese remainder theorem. In both cases,

$$(\mathbb{F}_2[g])^\times \simeq \mathbb{F}_{16}^\times \simeq \mathbb{Z}/15\mathbb{Z}.$$

Since $\mathbb{F}_2[g]$ is a subring of R, the group $(\mathbb{F}_2[g])^\times$ must be a subgroup of $R^\times$. However, $R^\times \simeq \mathbb{Z}/5\mathbb{Z}$ cannot contain a subgroup of order 15, which yields a contradiction.

To learn about the essential image of the functor $(-)^\times$: Ring → Group, the reader may consult Stackexchange and the references therein.

(7) The inclusion Ring → Rng is not full. A counterexample: $\mathrm{Hom}_{\mathsf{Ring}}(\mathbb{Z},\mathbb{Z}) = \{\mathrm{id}\}$, while $\mathrm{Hom}_{\mathsf{Rng}}(\mathbb{Z},\mathbb{Z}) \simeq \mathbb{Z}$.

(8) The inclusion Ring → Rng is not essentially surjective. A counterexample: The set $2\mathbb{Z}$ of even integers is closed under addition and multiplication and has an additive identity, 0, so it is a rng. But it is not a ring because it does not have a multiplicative identity. **Warning:** $\mathbb{Q}/\mathbb{Z}$ is not a rng if one use the "obvious" operation of just multiplying coset representatives. For example, $\frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$, while $\frac{3}{2} \times \frac{4}{3} = 2$. The elements $1/6$ and 2 represent distinct elements of $\mathbb{Q}/\mathbb{Z}$.

(9) The inclusion Field → Ring is not essentially surjective. A counterexample: $\mathbb{Z}$.

(10) The forgetful functor Mod_R → Ab is full if and only if the homomorphism $\mathbb{Z} \to R$ of rings is an epimorphism in the category of rings. We formulate a general result in Theorem 1, whose statement and proof will appear after this solution.

In our case, we want to know what epimorphisms $\mathbb{Z} \to R$ look like. The answer to this is known. In fact, these rings R and their classification seem to have been (re)invented several times, as "solid rings[4]" by Bousfield and Kan, as "T-rings" by R. A. Bowshell and P. Schultz... For more details, see Mathoverflow, where you can find a complete list of such rings.

Now, we give some examples and non-examples. It is easy to check that the homomorphisms from $\mathbb{Z}$ to its quotients and localizations are always epimorphisms. For example, $\mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Q}$ (an exercise we have met before). But $\mathbb{Z} \to \mathbb{C}$ is not an epimorphism. Or, $\mathsf{Mod}_{\mathbb{C}} \to \mathsf{Ab}$ is not full. The conjugation $\mathbb{C} \to \mathbb{C}$ is $\mathbb{Z}$-linear, but not $\mathbb{C}$-linear.

(11) The forgetful functor $\mathsf{Mod}_R \to \mathsf{Ab}$ is essentially surjective if and only if there exists a ring homomorphism $R \to \mathbb{Z}$. In fact,if this functor is essentially surjective, then there is an R-action on the abelian group $\mathbb{Z}$. In other words, there exists a ring homomorphism $R \to \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}) \simeq \mathbb{Z}$. Conversely, if there exists a ring homomorphism $\phi : R \to \mathbb{Z}$, then for any abelian group M, define the R-action by

$$r \cdot x := \phi(r)x \text{ (usual integer scaling).}$$

This makes M an R-module whose underlying abelian group is M itself[5].

Every homomorphism of rings $R \to \mathbb{Z}$ is automatically surjective, because the composition $\mathbb{Z} \to R \to \mathbb{Z}$ is the identity. There are many rings with a (surjective) ring homomorphism $R \to \mathbb{Z}$. For example, $\mathbb{Z}[X]/(X) \simeq \mathbb{Z}$. But fields do not have such property.

(12) The forgetful functor $\mathsf{Mod}_R \to \mathsf{Ab}$ is an equivalence if and only if $R \simeq \mathbb{Z}$. A functor is an equivalence if and only if it is full, faithful and essentially surjective. From the above, the homomorphism $i : \mathbb{Z} \to R$ is an epimorphism and there is a surjective homomorphism $p : R \to \mathbb{Z}$. Then $pi = \mathrm{id}_{\mathbb{Z}}$ and thus $ipi = i = \mathrm{id}_R \circ i$. Since i is an epimorphism, we get $ip = \mathrm{id}_R$. Hence $i : \mathbb{Z} \to R$ is an isomorphism with inverse p. □

Theorem 1. *Let $f : R \to S$ be a homomorphism of rings. Then f is an epimorphism in the category of rings if and ony if the induced functor $F : \mathsf{Mod}_S \to \mathsf{Mod}_R$ is full.*

Proof. **$\Longrightarrow$:** Let M and N be two S-modules, and let $\phi : M \to N$ be a homomorphism of R-modules. Let $g_1, g_2 : S \rightrightarrows \mathrm{End}_{\mathbb{Z}}(M \oplus N)$ be defined as follows:

$$g_1(s) = \begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix},$$

$$g_2(s) = \begin{pmatrix} s & 0 \\ \phi \circ s - s \circ \phi & s \end{pmatrix}.$$

Clearly, g_1 is a homomorphism of rings. To check g_2 is also a homomorphism of rings, we have to verify $g_2(st) = g_2(s)g_2(t)$, which can be obtained by directly computing the multiplication of matrices. Now, $g_1 \circ f = g_2 \circ f$ because ϕ is R-linear. Since f is an epimorphism, $g_1 = g_2$. Thus $\phi \circ s - s \circ \phi = 0$ for all $s \in S$, i.e., $\phi(sm) = s\phi(m)$. So ϕ is S-linear and F is full.

[4] This concept is distinct from the notion of a "solid" in condensed mathematics.

[5] In fact, this is a general fact: if $f : A \to B$ is a homomorphism of rings and M is a B-module, then we can define an A-action by $a \cdot m := f(a)m$, which makes M an A-module. Thus we get a functor $\mathsf{Mod}_B \to \mathsf{Mod}_A$. An interesting exercise is to find the left and right adjoints of this functor.

⟸: Let $g_1, g_2 : S \rightrightarrows T$ be two morphisms such that $g_1 f = g_2 f$. Then there are two ways to view T as an S-module: $s \cdot t := g_i(s)t$. Write these two S-modules as T_1, T_2 respectively (they are the same as a set, but they have distinct S-actions). The condition $g_1 f = g_2 f$ implies that T_1 and T_2 are the same as R-modules. In other words, id_T is R-linear. By fullness of F, id_T is S-linear, i.e., $g_1(s)t = g_2(s)t$ for any $s \in S$ and $t \in T$. In particular, taking $t = 1$, we obtain $g_1 = g_2$. Hence f is an epimorphism. □

Exercise (2.2.iv)**.** Prove the following strengthening of [Rie16, Lemma 1.2.3], demonstrating the equivalence between an isomorphism in a category and a representable isomorphism between the corresponding co- or contravariant represented functors: the following are equivalent:

(i) $f : x \to y$ is an isomorphism in C.
(ii) $f_* : \mathsf{C}(-, x) \Rightarrow \mathsf{C}(-, y)$ is a natural isomorphism.
(iii) $f^* : \mathsf{C}(y, -) \Rightarrow \mathsf{C}(x, -)$ is a natural isomorphism.

Proof. By the Yoneda Lemma [Rie16, Corollary 2.2.8], the Yoneda functors are fully faithful. Recall that [Rie16, Lemma 1.3.8] functors preserve isomorphisms and that [Rie16, Exercise 1.5.iv] a full and faithful functor reflects isomorphisms (we give a proof in the **Remark** below). Thus both (ii) and (iii) are equivalent to (i).

An alternative proof: Clearly, functors preserve isomorphisms. Thus (i) implies (ii) and (iii). Suppose that $f_* : \mathsf{C}(-, x) \Rightarrow \mathsf{C}(-, y)$ is a natural isomorphism. Then there exists a morphism $g \in \mathsf{C}(y, x)$ such that

$$fg = f_*(g) = \mathrm{id}_y.$$

Noting that the lower horizontal arrow f_* in the commutative diagram

$$\begin{array}{ccc} \mathsf{C}(y,x) & \xrightarrow{f_*} & \mathsf{C}(y,y) \\ {\scriptstyle f^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathsf{C}(x,x) & \xrightarrow[f_*]{} & \mathsf{C}(x,y) \end{array} \qquad \begin{array}{ccc} g & \longmapsto & \mathrm{id}_y \\ \downarrow & & \downarrow \\ gf & \longmapsto & f \end{array}$$

is an isomorphism, we obtain

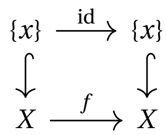
$$gf = \mathrm{id}_x.$$

Hence f is an isomorphism with inverse g. Similarly, we can use this method to show that (iii) implies (i). □

Remark. We give a proof of the fact that a full and faithful functor $F : \mathsf{C} \to \mathsf{D}$ reflects isomorphisms. Let $f : X \to Y$ be a morphism in C. Suppose that $F(f)$ is an isomorphism with inverse $g' : F(Y) \to F(X)$. Since F is full, there exists a morphism $g : Y \to X$ such that $F(g) = g'$. Then $F(fg) = F(f)F(g) = F(f)g' = \mathrm{id}_{F(Y)} = F(\mathrm{id}_Y)$ and $F(gf) = F(g)F(f) = g'F(f) = \mathrm{id}_{F(X)} = F(\mathrm{id}_X)$. Because F is faithful, we obtain that $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$, which means that f is an isomorphism with inverse g.

Exercise (2.2.vi)**.** Do there exist any non-identity natural endomorphisms of the category of spaces? That is, does there exist any family of continuous maps $X \to X$, defined for all spaces X and not all of which are identities, that are natural in all maps in the category Top?

Solution. No. We proceed by contradiction. Assume that such a family of continuous maps exists and that some $f : X \to X$ within it satisfies $f(x) \neq x$ for some $x \in X$. Note that the unique continuous map from the single-element space $\{x\}$ to itself is forced to be the identity. Thus the diagram

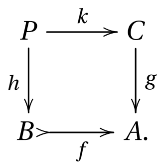


cannot be commutative, a contradiction. □

Exercise (3.1.v)**.** Consider a diagram $F : \mathsf{J} \to \mathsf{P}$ valued in a poset $(\mathsf{P}, \leq)$. Use order-theoretic language to characterize the limit and the colimit.

Proof. The limit is the infimum of $\{F(j)\}_{j\in \mathsf{J}}$, and the colimit is the supremum of $\{F(j)\}_{j\in \mathsf{J}}$. □

Exercise (3.1.vii)**.** Prove that if

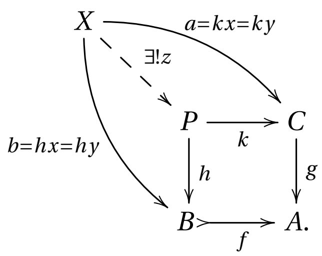


is a pullback square and f is a monomorphism, then k is a monomorphism.

Proof. Let $x, y : X \rightrightarrows P$ be two morphisms such that $kx = ky$. Then $fhx = gkx = gky = fhy$. Since f is a monomorphism, we obtain $hx = hy$. Denote by $a := kx = ky$ and $b := hx = hy$. Then

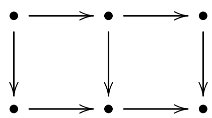
$$ga = fb.$$

By the universal property of pullback, there exists a unique $z : X \to P$ such that the following diagram is commutative



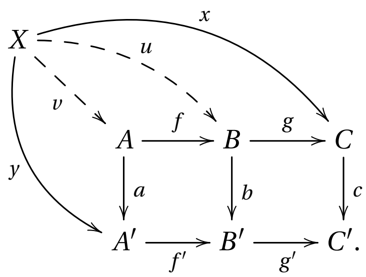
Since both x and y satisfy this universal property, we have $x = y$, which means that k is a monomorphism. □

Exercise (3.1.viii)**.** Consider a commutative rectangle



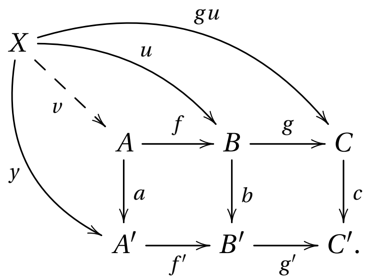
whose right-hand square is a pullback. Show that the left-hand square is a pullback if and only if the composite rectangle is a pullback.

Proof. $\Longrightarrow$**:** Suppose we have a commutative diagram of solid arrows in which both the left-hand and right-hand squares are pullbacks



The equality $cx = g' \circ (f'y)$ implies, by the universal property of the pullback (the right-hand square), the existence of a unique u with $x = gu$ and $f'y = bu$. Then the latter equality $f'y = bu$ implies, by the universal property of the pullback (the left-hand square), the existence of a unique v with $u = fv$ and $y = av$. Hence the composite rectangle is a pullback.

$\Longleftarrow$**:** Suppose we have a commutative diagram of solid arrows in which both the right-hand square and the composite rectangle are pullbacks



The equality $c(gu) = (g'f')y$ implies, by the universal property of the pullback (the composite rectangle), the existence of a unique v with $gu = (gf)v$ and $y = av$. It follows from the uniqueness part of the universal property of the right-hand pullback that $u = fv$. Hence the composite rectangle is a pullback. □

Exercise (3.1.ix)**.** Show that if J has an initial object, then the limit of any functor indexed by J is the value of that functor at an initial object. Apply the dual of this result to describe the colimit of a diagram indexed by a successor ordinal.

Proof. Let $F : \mathsf{J} \to \mathsf{C}$ be a functor. Let i be an initial object of J, i.e., for each $j \in \mathsf{J}$, there exists a unique morphism from $f_j : i \to j$. Suppose that there exist an object $X \in \mathsf{X}$ and a collection of morphisms $\{g_j : X \to F(j)\}_{j \in \mathsf{J}}$ such that for each morphism $\rho : j \to j'$ in J, we have $g_{j'} = F(\rho) \circ g_j$. In particular, $g_j = F(f_j) \circ g_i$ for any $j \in \mathsf{J}$. Assume $\phi : X \to F(i)$ to be another morphism such that $g_j = F(f_j) \circ \phi$. In particular, $g_i = F(f_i) \circ \phi$, i.e., $g_i = \phi$ as $f_i = \text{id}$. It means that $F(i)$ satisfies the universal property of limit. We are done.

The dual of this result: if J has a final object, then the colimit of any functor indexed by J is the value of that functor at a final object.

A diagram indexed by a successor ordinal $\alpha = \beta + 1$ is a functor from the category $(\beta + 1)$, viewed as a poset category, into some target category C. In simpler terms, it is a chain of objects and morphisms:

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots \longrightarrow X_\beta \xrightarrow{f_\beta} X_{\beta+1}.$$

This diagram has a final object, $X_\alpha = X_{\beta+1}$. The colimit of a diagram indexed by a successor ordinal $\alpha = \beta + 1$ is simply the final object in the diagram, X_α. □

Exercise. For any small category C, the presheaf category $\mathsf{PShv}(\mathsf{C}) := \mathsf{Func}(\mathsf{C}^{\mathrm{op}}, \mathsf{Set})$ is complete and co-complete. Hence the Yoneda embedding functor $Y : \mathsf{C} \to \mathsf{PShv}(\mathsf{C})$ could be regarded as a kind of "completion" for categorical calculus.

(a) the subcategory $Y(\mathsf{C})$ is "dense" in $\mathsf{PShv}(\mathsf{C})$, in the sense that, any object is indeed a colimit of objects from $Y(\mathsf{C})$.
(b) the functor Y is limit preserving.

Proof. (a) Let F be a functor from C^{op} to Set. Define a category D_F: objects are pairs (x, y) where $x \in \mathsf{C}$ and $y \in F(x)$, and morphisms $(x, y) \to (x', y')$ are morphisms $f : x \to x'$ such that $F(f)(y') = y$. Consider the projection functor $P : \mathsf{D}_F \to \mathsf{C}$ such that $P(x, y) = x$. We claim that

$$F \simeq \varinjlim_{\mathsf{D}_F} (Y \circ P).$$

Since colimits are computed point-wisely, it suffices to show that for each $x_0 \in \mathsf{C}$,

$$F(x_0) \simeq \varinjlim_{(x,y)\in \mathsf{D}_F} \mathrm{Mor}_{\mathsf{C}}(x_0, x).$$

First, we need a compatible collection of maps $\{\mathrm{Mor}_{\mathsf{C}}(x_0, x) \to F(x_0)\}$. For any pair $(x, y) \in \mathsf{D}_F$, we define

$$\begin{aligned} \alpha_{x,y} : \mathrm{Mor}_{\mathsf{C}}(x_0, x) &\longrightarrow F(x_0) \\ g &\longmapsto F(g)(y). \end{aligned}$$

Then for any morphism $f : (x, y) \to (x', y')$ in D_F, i.e., any morphism $f : x \to x'$ such that $F(f)(y') = y$, we have

$$F(fg)(y') = F(g)(F(f)(y')) = F(g)(y),$$

In other words,

$$\alpha_{x',y'} f_*(g) = \alpha_{x,y}(g),$$

i.e., the following diagram is commutative

$$\mathrm{Mor}_{\mathsf{C}}(x_0, x) \xrightarrow{f_*} \mathrm{Mor}_{\mathsf{C}}(x_0, x'), \quad \alpha_{x,y} : \mathrm{Mor}_{\mathsf{C}}(x_0, x) \to F(x_0), \quad \alpha_{x',y'} : \mathrm{Mor}_{\mathsf{C}}(x_0, x') \to F(x_0).$$

Now, we show that $F(x_0)$ satisfies the universal property of colimit. Suppose that S is a set and $\{\beta_{x,y} : \mathrm{Mor}_{\mathsf{C}}(x_0, x) \to S\}_{(x,y)\in \mathsf{D}_F}$ is a compatible collection of maps. We define

$$\begin{aligned} \gamma : F(x_0) &\longrightarrow S \\ y_0 &\longmapsto \beta_{x_0,y_0}(\mathrm{id}_{x_0}). \end{aligned}$$

Then for any pair $(x, y) \in \mathsf{D}_F$ and for any $g \in \mathrm{Mor}_{\mathsf{C}}(x_0, x)$, we have

$$\gamma(\alpha_{x,y}(g)) = \gamma(F(g)(y)) = \beta_{x_0,F(g)(y)}(\mathrm{id}_{x_0}) = (\beta_{x,y}g_*)(\mathrm{id}_{x_0}) = \beta_{x,y}(g),$$

where the third equality holds by the compatibility of the β's. Thus

$$\gamma\alpha_{x,y} = \beta_{x,y},$$

i.e., the following diagram is commutative

$$\begin{array}{ccc} & \mathrm{Mor}_{\mathsf{C}}(x_0, x) & \\ {\scriptstyle \alpha_{x,y}} \swarrow & & \searrow {\scriptstyle \beta_{x,y}} \\ F(x_0) & \xrightarrow[\gamma]{} & S. \end{array}$$

To prove uniqueness of such a γ, take $(x, y) = (x_0, y_0)$ and $g = \mathrm{id}_{x_0}$. Then the condition forces $\gamma(y_0)$ to be the element previously constructed, so any such γ must coincide with our definition.

(b) Let J be a small category and $F : \mathsf{J} \to \mathsf{C}$ be a functor. We wan to show a natural isomorphism

$$\mathrm{Mor}_{\mathsf{C}}(-, \varprojlim F) \simeq \varprojlim \mathrm{Mor}_{\mathsf{C}}(-, F(i)).$$

It suffices to show that for any morphism $f : x \to y$ in C, the left-hand square of the following diagram is commutative

$$\begin{array}{ccccc} \mathrm{Mor}_{\mathsf{C}}(y, \varprojlim F) & \longrightarrow & \varprojlim \mathrm{Mor}_{\mathsf{C}}(y, F(i)) & \longrightarrow & \mathrm{Mor}_{\mathsf{C}}(y, F(i)) \\ {\scriptstyle f^*}\downarrow & & \downarrow & & \downarrow{\scriptstyle f^*} \\ \mathrm{Mor}_{\mathsf{C}}(x, \varprojlim F) & \longrightarrow & \varprojlim \mathrm{Mor}_{\mathsf{C}}(x, F(i)) & \longrightarrow & \mathrm{Mor}_{\mathsf{C}}(x, F(i)), \end{array}$$

where the middle vertical arrow is obtained by applying the universal property of $\varprojlim \mathrm{Mor}_{\mathsf{C}}(x, F(i))$ to the compositions $\varprojlim \mathrm{Mor}_{\mathsf{C}}(y, F(i)) \to \mathrm{Mor}_{\mathsf{C}}(y, F(i)) \to \mathrm{Mor}_{\mathsf{C}}(x, F(i))$. In other words, the right-hand square is commutative for each $i \in \mathsf{J}$. Noting that the outer rectangle is commutative, we conclude that the left-hand square is commutative by the uniqueness part of the universal property of $\varprojlim \mathrm{Mor}_{\mathsf{C}}(x, F(i))$.

Alternative Proof: In the above proof, we used that colimits are computed pointwisely. Below, we give another proof without using this fact. Instead, we check that $\mathrm{Mor}_{\mathsf{C}}(-, \varprojlim F)$ satisfies the universal property of $\varprojlim \mathrm{Mor}_{\mathsf{C}}(-, F(i))$. Let $G : \mathsf{C}^{\mathrm{op}} \to \mathsf{Set}$ be a functor, and let $\{\alpha_i : G \to \mathrm{Mor}_{\mathsf{C}}(-, F(i))\}_{i \in \mathsf{J}}$ be a collection of natural transformations such that the following diagram is commutative

$$\begin{array}{ccc} & G & \\ {\scriptstyle \alpha_i} \swarrow & & \searrow {\scriptstyle \alpha_j} \\ \mathrm{Mor}_{\mathsf{C}}(-, F(i)) & \xrightarrow[F(\phi)_*]{} & \mathrm{Mor}_{\mathsf{C}}(-, F(j)) \end{array}$$

for any morphism $\phi : i \to j$. Then for any $x \in \mathsf{C}$ and for any $y \in G(x)$, we have a commutative diagram

$$\begin{tikzcd} & x \arrow[dl, "\alpha_{i,x}(y)"'] \arrow[dr, "\alpha_{j,x}(y)"] & \\ F(i) \arrow[rr, "F(\phi)"'] & & F(j). \end{tikzcd}$$

Thus there exists a unique morphism $\alpha_x(y) : x \to \varprojlim F$ such that $\alpha_{i,x}(y)$ factors through $\alpha_x(y)$ for each $i \in \mathsf{J}$. So we get a map $\alpha_x : G(x) \to \mathrm{Mor}_{\mathsf{C}}(x, \varprojlim F)$. Now, it remains to show that for any morphism $f : x \to x'$ in C, the following diagram is commutative

$$\begin{tikzcd} G(x') \arrow[r, "\alpha_{x'}"] \arrow[d, "G(f)"'] & \mathrm{Mor}_{\mathsf{C}}(x', \varprojlim F) \arrow[d, "f^*"] \\ G(x) \arrow[r, "\alpha_x"'] & \mathrm{Mor}_{\mathsf{C}}(x, \varprojlim F) \end{tikzcd}$$

In other words, we need to check that for any $y' \in G(x')$,

(4) $$\alpha_x(G(f)(y')) = \alpha_{x'}(y') \circ f.$$

Since $\alpha_i : G \to \mathrm{Mor}_{\mathsf{C}}(-, F(i))$ are natural transformations, we have

$$\alpha_{i,x}(G(f)(y')) = \alpha_{i,x}(y') \circ f.$$

Thus $\alpha_{x'}(y') \circ f$ is the unique morphism making the following diagram commutative

$$\begin{tikzcd} x \arrow[dd, "f"'] \arrow[drr, dashed] \arrow[drrr, "\alpha_{i,x}G(f)(y')"] & & & \\ & & \varprojlim F \arrow[r] & F(i) \\ x'. \arrow[urr, "\alpha_{x'}(y)"] \arrow[urrr, "\alpha_{i,x'}(y')"'] & & & \end{tikzcd}$$

So (4) holds. We are done. □

Remark. The first assertion of this exercise appears as [Rie16, Theorem 6.5.7]. The reader is encouraged to compare it with the proof provided above.

Exercise (4.1.i)**.** Consider a pair of functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ equipped with isomorphisms $\mathsf{D}(Fc, d) \cong \mathsf{C}(c, Gd)$ for each $c \in \mathsf{C}$ and $d \in \mathsf{D}$. Show that the naturality of this collection of isomorphisms is equivalent to the assertion that for any morphisms with domains and co-domains as displayed below

(5) $$\begin{tikzcd} Fc \arrow[r, "f^\sharp"] \arrow[d, "Fh"'] & d \arrow[d, "k"] & \rightsquigarrow & c \arrow[r, "f^\flat"] \arrow[d, "h"'] & Gd \arrow[d, "Gk"] \\ Fc' \arrow[r, "g^\sharp"'] & d' & & c' \arrow[r, "g^\flat"'] & Gd' \end{tikzcd}$$

the left-hand square commutes in D if and only if the right-hand transposed square commutes in C.

Proof. $\Longrightarrow$**:** Suppose that F is left adjoint to G. Then the following diagrams

$$\begin{array}{ccc} \mathsf{D}(Fc,d) & \xrightarrow{\sim} & \mathsf{C}(c,Gd) \\ {\scriptstyle k_*}\downarrow & & \downarrow{\scriptstyle (Gk)_*} \\ \mathsf{D}(Fc,d') & \xrightarrow{\sim} & \mathsf{C}(c,Gd') \end{array} \qquad \begin{array}{ccc} \mathsf{D}(Fc',d') & \xrightarrow{\sim} & \mathsf{C}(c',Gd') \\ {\scriptstyle (Fh)^*}\downarrow & & \downarrow{\scriptstyle h^*} \\ \mathsf{D}(Fc,d') & \xrightarrow{\sim} & \mathsf{C}(c,Gd') \end{array} \tag{6}$$

are commutative, i.e.,

$$(k\circ f^\sharp)^\flat = Gk\circ f^\flat, \quad (g^\sharp\circ Fh)^\flat = g^\flat\circ h.$$

Thus $k\circ f^\sharp = g^\sharp\circ Fh$ if and only if $Gk\circ f^\flat = g^\flat\circ h$. In other words, the commutativity of the two squares in (5) are equivalent to each other.

$\Longleftarrow$**:** In (5), taking $c' = c$, $h = \mathrm{id}_c$, we see that $g^\sharp = kf^\sharp$ and $g^\flat = Gk\circ g^\flat$ are the corresponding morphisms under the isomorphism $\mathsf{D}(Fc,d') \simeq \mathsf{C}(c,Gd')$. It means that the first square in (6) is commutative. Similarly, the second square in (6) is commutative. Hence F is adjoint to G. ☐

Exercise (4.1.ii)**.** Define left and right adjoints to

(i) $\mathrm{ob}:\mathsf{Cat}\to\mathsf{Set}$,
(ii) $\mathsf{Vert}:\mathsf{Graph}\to\mathsf{Set}$,
(iii) $\mathsf{Vert}:\mathsf{DirGraph}\to\mathsf{Set}$.

Proof. (i) The functor $\mathrm{ob}:\mathsf{Cat}\to\mathsf{Set}$ has a left adjoint which assigns to each set X the discrete category on X, and has a right adjoint which assigns to each set X a category with objects X and exactly one arrow in every hom-set.

(ii) The functor $\mathsf{Vert}:\mathsf{Graph}\to\mathsf{Set}$ has a left adjoint which assigns to a set X the graph with vertex set X and no edges, and has a right adjoint which assigns to each set X a graph with vertex set X and exactly one edge between any two vertices (and on functions it does nothing).

(iii) The functor $\mathsf{Vert}:\mathsf{DirGraph}\to\mathsf{Set}$ has a left adjoint which assigns to a set X the graph with vertex set X and no edges, and has a right adjoint which assigns to each set X a graph with vertex set X and exactly one arrow from one vertex to the another vertex (so two arrows with opposite directions between two distinct vertices). ☐

Exercise (4.1.iii)**.** Show that any triple of adjoint functors

$$\mathsf{C} \underset{R}{\overset{L}{\leftarrow}}\!\!\xrightarrow{\ U\ }\!\! \mathsf{D}, \qquad L\dashv U\dashv R$$

gives rise to a canonical adjunction $LU \dashv RU$ between the induced endo-functors of C.

Proof. For any $f: X\to X'$ and $g: Y\to Y'$ in C, we have the following two commutative diagrams

$$\begin{array}{ccccc} \mathsf{C}(LUX,Y) & \xrightarrow{\sim} & \mathsf{D}(UX,UY) & \xrightarrow{\sim} & \mathsf{C}(X,RUY) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle (Ug)_*} & & \downarrow{\scriptstyle (RUg)_*} \\ \mathsf{C}(LUX,Y') & \xrightarrow{\sim} & \mathsf{D}(UX,UY') & \xrightarrow{\sim} & \mathsf{C}(X,RUY'), \end{array}$$

and

$$\begin{array}{ccccc} \mathsf{C}(LUX', Y) & \xrightarrow{\sim} & \mathsf{D}(UX', UY) & \xrightarrow{\sim} & \mathsf{C}(X', RUY) \\ \downarrow{\scriptstyle (LUf)^*} & & \downarrow{\scriptstyle (Uf)^*} & & \downarrow{\scriptstyle f^*} \\ \mathsf{C}(LUX, Y) & \xrightarrow{\sim} & \mathsf{D}(UX, UY) & \xrightarrow{\sim} & \mathsf{C}(X, RUY), \end{array}$$

where the left-hand (resp. right-hand) squares are commutative because L (resp. R) is left (resp. right) adjoint to U. In conclusion, we have isomorphisms

$$\mathsf{C}(LUX, Y) \simeq \mathsf{C}(X, RUY)$$

for each $X, Y \in \mathsf{C}$ that is natural in both variables, which means that LU is left adjoint to RU. □

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