A SPECIALISATION THEOREM FOR LANG-NÉRON GROUPS

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ABSTRACT. We show that, for a polarised smooth projective variety $B \hookrightarrow \mathbb{P}^n_k$ of dimension ≥ 2 over an infinite field k and an abelian variety A over the function field of B, there exists a dense Zariski open set of smooth geometrically connected hyperplane sections h of B such that A has good reduction at h and the specialisation homomorphism of Lang-Néron groups at h is injective (up to a finite p-group in positive characteristic p). This gives a positive answer to a conjecture of the first author, which is used to deduce a negative definiteness result on his refined height pairing. This also sheds a new light on Néron's specialisation theorem.

CONTENTS

1. INTRODUCTION

Let K/k be a finitely generated regular extension of fields of exponential characteristic p, and let A be an abelian variety over K. Then A has a K/k-trace $C = \mathrm{Tr}_{K/k} A$: this operation is right adjoint to the extension of scalars of abelian varieties. A celebrated theorem of Lang and Néron says that the *Lang-Néron group*

$$\mathrm{LN}(K/k, A) = A(K)/C(k)$$

is finitely generated ([LN59], see also [Con06] and [Kah09]).

Let B be a smooth model of K/k, and let $h \in B$ be a point of codimension 1 whose residue field E is also regular over k. If A has good reduction at h, write A_h for its reduction and $C_h = \mathrm{Tr}_{E/k}(A_h)$. Then we have a k-morphism

$$\varphi_0 : C \longrightarrow C_h \tag{1}$$

and a commutative diagram of specialisation maps with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & C(k) \arrow[r] \arrow[d, "\varphi_0(k)"] & A(K) \arrow[r] \arrow[d, "\varphi"] & \mathrm{LN}(A, K/k) \arrow[r] \arrow[d, "\psi"] & 0 \\ 0 \arrow[r] & C_h(k) \arrow[r] & A_h(E) \arrow[r] & \mathrm{LN}(A_h, E/k) \arrow[r] & 0 \end{tikzcd} \tag{2}$$

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see [Kah24, §6B]. Recall their construction: the stalk $R := \mathscr{O}_{B,h}$ is a discrete valuation ring with field of fractions K and, by hypothesis, A has a smooth proper model $\mathscr{A}$ over R. Then $\mathscr{A}$ is automatically an abelian scheme and is the Néron model of A by [BLR90, §1.4, Proposition 2]. Then A_h is the special fibre of $\mathscr{A}$, which is an abelian variety over E; this yields φ. The co-unit $C_K \to A$ extends to a morphism $C_R \to \mathscr{A}$, and induces a morphism $C_E \to A_h$, thus the existence of φ_0 follows from the universal property of C_h.

Theorem 1.1. *Assume that B is smooth projective of dimension $d \geq 2$. For any projective embedding $B \hookrightarrow \mathbb{P}^n_k$, there exists a dense open subset $\mathcal{U}$ of the dual projective space $\mathcal{P}$ of $\mathbb{P}^n_k$ such that if H lies in $\mathcal{U}(k)$, then*

(a) *the hyperplane section $h := H \cap B$ is smooth geometrically connected of dimension $d-1$,*
(b) *A has good reduction at h,*
(c) *in Diagram (2), the kernels of all vertical maps are finite p-groups (hence these maps are injective in characteristic 0),*
(d) *the map φ_0 of (1) is a p-isogeny (an isomorphism in characteristic 0).*

(If k is infinite, so is $\mathcal{U}(k)$. When k is finite, $\mathcal{U}(k)$ may be empty because in general there are no smooth hyperplane sections in B defined over k; this issue can presumably be solved by composing the given projective embedding with a suitable Veronese embedding (see [Gab01, Corollary 1.6] and [Poo04, Theorem 3.1]).)

Besides Bertini's theorem, our main tool is a form of the weak Lefschetz theorem due to Deligne [Kat93, A.5], which renders the proof almost trivial.

The first application is to a negative definiteness result for the height pairing introduced in [Kah24]. For a smooth projective variety X of dimension d over K and $i \in [0, d]$, the first author defined a subgroup $\mathrm{CH}^i(X)^{(0)}$ of the i-th Chow group of X and a pairing

$$\mathrm{CH}^i(X)^{(0)} \times \mathrm{CH}^{d+1-i}(X)^{(0)} \to \mathrm{CH}^1(B) \otimes \mathbb{Q}.$$

For $i = 1$, this pairing induces a quadratic form on the Lang-Néron group of the Picard variety of X. In [Kah24, Theorem 6.6], it is proven that this quadratic form is negative definite if B is a curve, and that one can reduce to this case when $\dim B > 1$ if ψ has finite kernel in (2) for a suitable h [Kah24, Conjecture 6.3]. Thus Theorem 1.1 proves this conjecture[1] (in a stronger form, and without the hypothesis of semi-stable reduction appearing in loc. cit.).

The second application is to Néron's specialisation theorem. Assume that k is a number field. If $B = \mathbb{P}^n_k$ and U is an open subset of B over which A extends to an abelian scheme $\mathscr{A}$, then the set of rational points $t \in U(k)$ such that the specialisation map $A(K) \to \mathscr{A}_t(k)$ is not injective is thin ([Ser97, 11.1, theorem], see [CT20] for generalisations). The injectivity of φ in Theorem 1.1 gives a version of this specialisation result which does not involve Hilbert's irreducibility theorem, but of course requires $\dim B > 1$; see however Remark 3.4.

2. AUXILIARY RESULTS

We start with the following standard lemmas.

[1] At least for k infinite, but this is sufficient for the application: see [Kah24, part (d) of the proof of Theorem 6.6].

Lemma 2.1. *Let U be an integral normal noetherian scheme with function field K. Let $\mathscr{A}$ be an abelian scheme over U with generic fibre A. Then the pull-back map*

$$\mathscr{A}(U) \to A(K)$$

is an isomorphism.

Proof. This is a consequence of the valuative criterion of properness and Weil's extension theorem ([Art86, Proposition 1.3] or [BLR90, §4.4, Theorem 1]). □

Lemma 2.2. *Let U be a scheme and let $\mathscr{A}$ be an abelian scheme over U. If n is invertible on U, i.e., n is prime to* $\mathrm{char}(k(x))$ *for all $x \in U$, then we have an injection*

$$\mathscr{A}(U)/n \hookrightarrow H^1_{\text{ét}}(U, {}_n\mathscr{A}),$$

where ${}_n\mathscr{A}$ is the kernel of the multiplication by n on $\mathscr{A}$.

Proof. Use the short exact sequence of étale sheaves

$$0 \to {}_n\mathscr{A} \to \mathscr{A} \xrightarrow{n} \mathscr{A} \to 0.$$ □

By the above lemmas, we can use cohomology to study the specialisation of $A(K)$. We shall rely on the following version of the weak Lefschetz theorem.

Theorem 2.3 (Deligne; see [Kat93, Corollary A.5])**.** *Let k be a separably closed field and let $\ell \neq$* $\mathrm{char}(k)$ *be a prime. Let $f\colon U \to \mathbb{P}^n_k$ be a separated quasi-finite morphism and let $\mathscr{F}$ be a lisse $\overline{\mathbb{Q}_\ell}$-sheaf on U. Assume that U is smooth over k and is of pure dimension d. Then there exists a dense open subset $\mathcal{U}$ of the dual projective space $\mathcal{P}$ of $\mathbb{P}^n_k$ such that if H lies in $\mathcal{U}$, then the restriction map*

$$H^i(U, \mathscr{F}) \longrightarrow H^i(f^{-1}(H), \mathscr{F}|_{f^{-1}(H)})$$

is an isomorphism for $i < d-1$ and injective for $i = d-1$.

Proof. In fact, in loc. cit., this theorem is proven when k is algebraically closed for general perverse sheaves without assuming that U is smooth and is of pure dimension d. In our case $\mathscr{F}[d]$ is a perverse sheaf; see [KW01, p. 139]. Moreover, the algebraically closed case immediately implies the separably closed case. □

3. Proof of Theorem 1.1

Choose a dense open subset U of B and an abelian scheme $\mathscr{A}$ over U such that $A \simeq \mathscr{A}_K$ (see [Mil86, Remark 20.9]). Applying Bertini's theorem [Jou83, Corollary 6.11(2)] to B and U, we get a dense open subset $\mathcal{U}_1$ of the dual projective space $\mathcal{P}$ of $\mathbb{P}^n_k$ such that if H lies in $\mathcal{U}_1(k)$ then $B \cap H$ (hence $U \cap H$) is smooth and geometrically connected of dimension $d-1$, and $U \cap H \neq \emptyset$. In particular, A has good reduction at $B \cap H$ if $H \in \mathcal{U}_1(k)$.

Let ℓ be a prime different from p. Then by [BLR90, §7.3, Lemma 2], the kernel ${}_{\ell^m}\mathscr{A}$ of multiplication by ℓ^m on $\mathscr{A}$ is finite and étale. Thus it represents a locally constant constructible étale sheaf on U. Denote by $T_\ell\mathscr{A}$ the lisse ℓ-adic sheaf $({}_{\ell^m}\mathscr{A})$.

Let k_s be a separable closure of k. We denote base change from k to k_s by an index s. Note that the immersion $f\colon U \hookrightarrow \mathbb{P}^n_k$ induced by the projective embedding $B \hookrightarrow \mathbb{P}^n_k$ is separated quasi-finite. By Theorem 2.3, there exists a dense open subset $\mathcal{U}_2$ of the dual projective space $\mathcal{P}_s$ such that if H lies in $\mathcal{U}_2$, then the restriction map

$$H^i(U_s, T_\ell\mathscr{A}) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}_\ell} \longrightarrow H^i(U_s \cap H, T_\ell\mathscr{A}) \otimes_{\mathbb{Z}_\ell} \overline{\mathbb{Q}_\ell}$$

is an isomorphism for $i < d-1$ and injective for $i = d-1$. Therefore the restriction map

$$H^i(U_s, T_\ell\mathscr{A}) \longrightarrow H^i(U_s \cap H, T_\ell\mathscr{A})$$

has finite kernel and cokernel for $i < d-1$ and finite kernel for $i = d-1$. (Recall that $H^i_{\text{ét}}(U_s, {}_{\ell^m}\mathscr{A})$ is finite for all m by [SGA $4\frac{1}{2}$, Th. finitude], hence $H^i(U_s, T_\ell\mathscr{A})$ is a finitely generated $\mathbb{Z}_\ell$-module.)

The open subset $\mathcal{U}_2$ is defined over a finite Galois extension of k; taking the intersection of its conjugates, we may assume that it is defined over k. Take $\mathcal{U} = \mathcal{U}_1 \cap \mathcal{U}_2$.

We now proceed in three steps:

3.1. $\operatorname{Ker}\psi$ **is finite.** For $H \in \mathcal{U}(k)$, we write $h = B \cap H$. Since the groups $C(k_s)$ and $C_h(k_s)$ are ℓ-divisible, we have the isomorphisms

$$\mathscr{A}(U_s)/\ell^m \simeq (\mathscr{A}(U_s)/C(k_s)) \otimes_{\mathbb{Z}} \mathbb{Z}/\ell^m\mathbb{Z} \simeq \mathrm{LN}(A, Kk_s/k_s)/\ell^m,$$

where the second one holds by Lemma 2.1. Similarly, we have such isomorphisms for $\mathrm{LN}(A_h, Ek_s/k_s)$. Taking the inverse limit of the following commutative diagrams

$$\begin{tikzcd}
H^1_{\text{ét}}(U_s, {}_{\ell^m}\mathscr{A}) \arrow[d] & \mathscr{A}(U_s)/\ell^m \arrow[l, hook'] \arrow[r, "\sim"] \arrow[d] & \mathrm{LN}(A, Kk_s/k_s)/\ell^m \arrow[d] \\
H^1_{\text{ét}}((U\cap h)_s, {}_{\ell^m}\mathscr{A}) & \mathscr{A}((U\cap h)_s)/\ell^m \arrow[l, hook'] \arrow[r, "\sim"] & \mathrm{LN}(A_h, Ek_s/k_s)/\ell^m
\end{tikzcd}$$

we get the commutative diagram

$$\begin{tikzcd}
H^1(U_s, T_\ell\mathscr{A}) \arrow[d] & \mathscr{A}(U_s)^\wedge \arrow[l, hook'] \arrow[r, "\sim"] \arrow[d] & \mathrm{LN}(A, Kk_s/k_s)^\wedge \arrow[d, "\psi^\wedge"] \\
H^1((U\cap h)_s, T_\ell\mathscr{A}) & \mathscr{A}((U\cap h)_s)^\wedge \arrow[l, hook'] \arrow[r, "\sim"] & \mathrm{LN}(A_h, Ek_s/k_s)^\wedge
\end{tikzcd}$$

where $(-)^\wedge$ denotes ℓ-adic completion. Since the left vertical arrow has finite kernel, so do the others. By the Lang-Néron theorem, the abelian group $\mathrm{LN}(A, K/k)$ is finitely generated. Thus $\psi^\wedge = \psi \otimes \mathbb{Z}_\ell$, which implies that ψ has a finite kernel. But $\mathrm{LN}(A, K/k)$ injects into $\mathrm{LN}(A, Kk_s/k_s)$, so we are done.

3.2. **The map φ_0 is an isogeny.** Given an abelian group M, we have the ℓ-adic Tate module of M defined by

$$T_\ell M = \varprojlim_m ({}_{\ell^m}M),$$

where ${}_{\ell^m}M$ is the ℓ^m-torsion of M and the inverse limit is over positive integers m with transition morphisms given by the multiplication-by-ℓ map ${}_{\ell^{m+1}}M \to {}_{\ell^m}M$. The Tate modules of the Lang-Néron groups vanish because the Lang-Néron groups are finitely generated and the transition map is multiplication by ℓ. Consider the commutative diagram

$$\begin{tikzcd}
T_\ell(C(k_s)) \arrow[r] \arrow[d, "T_\ell(\varphi_0)"] & T_\ell(A(Kk_s)) \arrow[d, "T_\ell(\varphi)"] & H^0(U_s, T_\ell\mathscr{A}) \arrow[l] \arrow[d] \\
T_\ell(C_h(k_s)) \arrow[r] & T_\ell(A_h(Ek_s)) & H^0((U\cap h)_s, T_\ell\mathscr{A}). \arrow[l]
\end{tikzcd}$$

Applying the (left exact) Tate module functor to Diagram (2), we see that the left horizontal maps are isomorphisms; Lemma 2.1 then shows the same for the right horizontal maps.

Applying Theorem 2.3 to the right vertical arrow, we obtain that all vertical arrows have finite kernels and cokernels. By [Mil86, Remark 8.4], $T_\ell(C(k_s))$ (resp. $T_\ell(C_h(k_s))$) is a free $\mathbb{Z}_\ell$-module whose rank is $2\dim C$ (resp. $2\dim C_h$). It follows that the left vertical arrow is injective, and that C and C_h have the same dimension. The injectivity of $T_\ell(\varphi_0)$ implies that the ℓ-adic Tate module of the abelian variety $(\ker\varphi_0)^0_{\mathrm{red}}$ vanishes. Thus $(\ker\varphi_0)^0_{\mathrm{red}}$ is a 0-dimensional abelian variety and $\ker\varphi_0$ is a finite group scheme. Hence φ_0 is an isogeny.

3.3. **End of proof.** So far, we have proven that the kernels of the vertical maps in (2) are finite, and it remains to show that they are p-primary. Since the formation of K/k-trace commutes with base change [Con06, Th. 6.8] and $\mathrm{LN}(A, K/k)$ injects into $\mathrm{LN}(A, Kk_s/k_s)$ (ibid., proof of Lemma 7.3), we reduce to the case $k = k_s$. But then, the map $\varphi\colon A(K) \to A_h(E)$ is injective on n-torsion for any n invertible in k [Ser97, p. 153] and $C_h(k)$ is n-divisible for such n [Mil86, Th. 8.2]. Thus the conclusion follows by applying the snake lemma to Diagram (2).

3.4. **Remark.** Let x be a closed point of B, let A_x be the special fiber of $\mathscr{A}$ at x, and let R_x be the Weil restriction of A_x through the finite extension $k(x)/k$. The map $C_U \to \mathscr{A}$ induces a map $C_{k(x)} \to A_x$ and then induces a map $C \to R_x$ by functoriality. Consider the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & C(k) \arrow[r] \arrow[d, "\varphi_0"] & A(K) \arrow[r] \arrow[d, "\varphi"] & \mathrm{LN}(A, K/k) \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & R_x(k) \arrow[r, "\sim"] & A_x(k(x)) \arrow[r] & 0 \arrow[r] & 0.
\end{tikzcd}$$

The snake lemma gives us an exact sequence

$$0 \to \operatorname{Ker}\varphi_0 \to \operatorname{Ker}\varphi \to \mathrm{LN}(A, K/k) \to A_x(k(x))/C(k)$$

A similar argument to §3.3 shows that $\operatorname{Ker}\varphi_0$ is a finite p-group. Thus $\operatorname{Ker}\varphi$ is finitely generated and its rank is uniformly bounded when x varies.

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