EXTREME VALUES OF QUADRATIC HECKE L-FUNCTIONS OVER GLOBAL FIELDS

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Abstract. We study large values of quadratic Hecke L-functions in the conductor aspect. Let K be a fixed number field, and assume GRH for its finite-order Hecke L-functions. In a fixed ray class component with conductor norm comparable to X, we prove that

$$\max_{\chi} L\left(\frac{1}{2}+\frac{A}{\log_2 X},\chi\right) \geq \exp\left\{(e^{-A}+o(1))\sqrt{\frac{\log X\log_3 X}{\log_2 X}}\right\}$$

for every fixed $A \geq 0$. Every fixed smaller constant is attained by at least $X^{1-o(1)}$ characters. The same count holds at a suitably slowly moving threshold approaching the displayed constant. The combinatorial input is a sparse squarefree Gál set of cardinality N, retaining the known leading constant 2 and having square multiplicative energy $N^{2+o(1)}$. The energy bound reflects the low degree of the associated Boolean polynomial. Together with the resonance estimate, it yields the abundance bound. We also prove unconditional analogues for quadratic characters with prime conductor away from one fixed place over any global function field of odd characteristic. Finally, we give bounds in the fixed strip and at $s = 1$, including the dependence on the residue of the Dedekind zeta function and the prescribed local factors.

Contents

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1. Introduction

We study large values of quadratic Hecke L-functions near the central point, with particular attention to how often such values occur. A lower bound for the maximum obtained by resonance need not, by itself, give a useful count of the characters attaining large values: the resonating weight may be concentrated on too few members of the family. Here we use sparse squarefree resonators whose fourth moments can also be controlled. This allows us to prove that every fixed lower threshold in our maximum estimate is reached by $X^{1-o(1)}$ characters in a family of size comparable to X. The argument applies at the centre and at real points approaching it from the right.

Let K be a fixed number field of degree d_K, with ring of integers $\mathcal{O}_K$, and let $\mathrm{N}\mathfrak{a}$ denote the absolute norm of a nonzero integral ideal. Unless stated otherwise, ideal sums are over nonzero integral ideals of $\mathcal{O}_K$. We write $\log_j$ for the j-fold iterated natural logarithm and set

$$\mathcal{V}(X) = \sqrt{\frac{\log X \log_3 X}{\log_2 X}}.$$

The family is chosen in Proposition 2.1. It consists of squarefree ideals $\mathfrak{a}$ in the identity class of a fixed narrow ray modulus with finite part $\mathfrak{q}$. Denote this set by Ω and the finite support of $\mathfrak{q}$ by S. For each $\mathfrak{a} \in \Omega$, choose a totally positive generator $\alpha_{\mathfrak{a}} \equiv 1 \pmod{\mathfrak{q}}$. With a suitable fixed totally positive $d \in K^{\times}$, the extension $K(\sqrt{d\alpha_{\mathfrak{a}}})/K$ defines a quadratic Hecke character $\chi_{\mathfrak{a}}$ independent of this choice. Its conductor satisfies

$$\mathfrak{f}_{\mathfrak{a}} = \mathfrak{c}_{\Omega}\mathfrak{a}, \qquad \chi_{\mathfrak{a}}(\mathfrak{p}) = 0 \quad (\mathfrak{p} \in S),$$

where $\mathfrak{c}_{\Omega}$ is fixed. As usual, the ideal coefficients are extended by zero at ramified primes. Thus the conductor norm is comparable to $\mathrm{N}\mathfrak{a}$. We write

$$\Omega(X) = \{\mathfrak{a} \in \Omega : X < \mathrm{N}\mathfrak{a} \le 2X\}.$$

Then $\#\Omega(X) \sim \kappa_{\Omega} X$ for a constant $\kappa_{\Omega} > 0$.

Throughout the number field arguments, GRH_K denotes GRH for all finite-order Hecke L-functions over K. The larger class of characters is needed in the reciprocity argument: extending a quadratic character from a subgroup of a ray class group need not preserve its order. For $A \ge 0$, put

$$\sigma_A(X) = \frac{1}{2} + \frac{A}{\log_2 X}. \tag{1.1}$$

Theorem 1.1. *Assume* GRH_K. *Then the following assertions hold.*

(i) *For every fixed* $A \ge 0$,
$$\max_{\mathfrak{a} \in \Omega(X)} L(\sigma_A(X), \chi_{\mathfrak{a}}) \ge \exp\{(e^{-A} + o(1))\mathcal{V}(X)\}.$$

(ii) *For every fixed* $A \ge 0$ *and* $0 < c < 1$,
$$\#\{\mathfrak{a} \in \Omega(X) : L(\sigma_A(X), \chi_{\mathfrak{a}}) \ge e^{ce^{-A}\mathcal{V}(X)}\} \ge X^{1-o(1)}.$$

(iii) *For every fixed* $A_0 \ge 0$, *there is a function* $\eta_{A_0}(X) \to 0$ *such that, uniformly for* $0 \le A \le A_0$,
$$\#\{\mathfrak{a} \in \Omega(X) : L(\sigma_A(X), \chi_{\mathfrak{a}}) \ge e^{(1-\eta_{A_0}(X))e^{-A}\mathcal{V}(X)}\} \ge X^{1-o(1)}.$$

The first two assertions are also uniform for A in a fixed compact interval.

At $A = 0$ this gives the corresponding statements at the centre. The count $X^{1-o(1)}$ means that, for each fixed $\varepsilon > 0$, there are at least $X^{1-\varepsilon}$ such characters for all sufficiently large X. This does not imply a positive proportion. The threshold in (iii) is obtained by diagonal selection, with no prescribed rate for $\eta_{A_0}(X) \to 0$.

Over $\mathbb{Q}$ the result can be stated in a single congruence class of positive fundamental discriminants. Write $\chi_D(n) = (D/n)$ for the Kronecker character associated with a positive fundamental discriminant D, and put

$$\mathcal{D}_8(X) = \{D : X < D \le 2X,\ D = 8a,\ a \text{ squarefree}, \quad a \equiv 1 \pmod 8\}.$$

These are precisely the positive fundamental discriminants in the congruence class $8 \pmod{64}$ and the indicated interval.

Corollary 1.2. *Assume GRH for Dirichlet L-functions. For every fixed $A \ge 0$ and $0 < c < 1$,*

$$\#\{D \in \mathcal{D}_8(X) : L(\sigma_A(X), \chi_D) \ge e^{ce^{-A}\mathcal{V}(X)}\} \ge X^{1-o(1)}.$$

Moreover,

$$\max_{D \in \mathcal{D}_8(X)} L(\sigma_A(X), \chi_D) \ge \exp\{(e^{-A} + o(1))\mathcal{V}(X)\}.$$

Both assertions are uniform for A in a fixed compact subset of $[0, \infty)$, and remain valid with $\mathcal{D}_8(X)$ replaced by all positive fundamental discriminants in $(X, 2X]$.

Proof. For $K = \mathbb{Q}$, take $S = \{2\}$, $d = 2$, and $\mathfrak{q} = (8)$ in the family construction. The component consists of squarefree positive integers $a \equiv 1 \pmod 8$. The extension $\mathbb{Q}(\sqrt{2a})$ has discriminant and conductor $8a$, and its character is χ_{8a}. To express the result in the conductor variable, set

$$Y = X/8, \qquad B = A\frac{\log_2 Y}{\log_2 X}.$$

Then $\sigma_B(Y) = \sigma_A(X)$ exactly, and, uniformly for bounded A,

$$e^{-B}\mathcal{V}(Y) = (1 + o(1))e^{-A}\mathcal{V}(X).$$

Apply Theorem 1.1(ii) with any fixed $c < c' < 1$ and use its compact-parameter uniformity. For sufficiently large X, the threshold with c' exceeds the one stated here, and $Y^{1-o(1)} = X^{1-o(1)}$. The maximum follows from assertion (i) of the same theorem. Inclusion gives the claims for all positive fundamental discriminants. □

The analogous result over function fields is unconditional. Let $F = \mathbb{F}_q(C)$, where q is odd and C is a smooth, projective, geometrically connected curve, and fix an odd-degree place ∞. We use the component Ω_F of principal primes in $\Gamma(C \setminus \{\infty\}, \mathcal{O}_C)$ constructed in Section 7. A normalized generator of $\mathfrak{P} \in \Omega_F$ defines a quadratic character $\chi_{\mathfrak{P}}$ with conductor divisor $\mathfrak{P} + \infty$. The finite conductor away from ∞ is therefore prime. For the admissible degrees, write

$$\Omega_F(n) = \{\mathfrak{P} \in \Omega_F : \deg \mathfrak{P} = n\}, \qquad X_n = q^n,$$

and set

$$\mathcal{V}_F(n) = \sqrt{\frac{n \log q \, \log_2 n}{\log n}} = (1 + o(1))\mathcal{V}(X_n), \tag{1.2}$$

$$\sigma_{A,n} = \frac{1}{2} + \frac{A}{\log_2 X_n}. \tag{1.3}$$

Theorem 1.3. *Fix $F = \mathbb{F}_q(C)$ with q odd, and let n tend to infinity through the admissible degrees of Ω_F.*

(i) *For every fixed $A \ge 0$,*

$$\max_{\mathfrak{P} \in \Omega_F(n)} L(\sigma_{A,n}, \chi_{\mathfrak{P}}) \ge \exp\{(e^{-A} + o(1))\mathcal{V}_F(n)\}.$$

(ii) *For every fixed $A \ge 0$ and $0 < c < 1$,*

$$\#\{\mathfrak{P} \in \Omega_F(n) : L(\sigma_{A,n}, \chi_{\mathfrak{P}}) \ge e^{ce^{-A}\mathcal{V}_F(n)}\} \ge X_n^{1-o(1)}.$$

(iii) *For every fixed $A_0 \geq 0$, there is a function $\eta_{F,A_0}(n) \to 0$ such that, uniformly for $0 \leq A \leq A_0$,*

$$\#\{\mathfrak{P} \in \Omega_F(n) : L(\sigma_{A,n}, \chi_{\mathfrak{P}}) \geq e^{(1-\eta_{F,A_0}(n))e^{-A}\mathcal{V}_F(n)}\} \geq X_n^{1-o(1)}.$$

The first two assertions are uniform on fixed compact intervals of A. No unproved hypothesis is required.

The restriction to prime conductors gives a twisted character average with error $O_F((1+\deg \mathfrak{l}_0)q^{n/2}/n)$, where $\mathfrak{l}_0$ is the squarefree part of the twisting ideal. Its linear dependence on $\deg \mathfrak{l}_0$ is useful here, since the resonator contains divisors of degree much larger than n.

We also obtain number field bounds at fixed points to the right of the centre. For $0 < b < 1$ and $1/2 < \sigma < 1$, put

$$\alpha(b) = 2\log\frac{(1+b)^2}{1+b^2}, \qquad \widetilde{\alpha}(\sigma, b) = \frac{2b}{(1+b^2)(1-\sigma)}\alpha(b)^{\sigma-1}. \tag{1.4}$$

The factor $2b/(1+b^2)$ comes from the local square product count. $\alpha(b)$ determines the permitted length of the resonator.

Theorem 1.4. *Assume* GRH_K*, and fix $1/2 < \sigma < 1$ and $0 < b < 1$.*

(i) *One has*

$$\max_{\mathfrak{a}\in\Omega(X)} \log L(\sigma, \chi_{\mathfrak{a}}) \geq (\widetilde{\alpha}(\sigma, b) + o(1))(\log X)^{1-\sigma}(\log_2 X)^{-\sigma}.$$

(ii) *For $0 < \eta < 1/\alpha(b)$, define*

$$\tau_{\sigma,\eta}(X) = \widetilde{\alpha}(\sigma, b)(1-\eta\alpha(b))^{1-\sigma}(\log X)^{1-\sigma}(\log_2 X)^{-\sigma}.$$

Then

$$\frac{\#\{\mathfrak{a} \in \Omega(X) : \log L(\sigma, \chi_{\mathfrak{a}}) > \tau_{\sigma,\eta}(X)\}}{\#\Omega(X)} \geq X^{-\frac{1}{2}(1-\eta\alpha(b))+o(1)}.$$

At $s = 1$ the leading multiplier includes the residue $\kappa_K = \mathrm{Res}_{s=1}\,\zeta_K(s)$ and the factors at the fixed ramified primes. With γ denoting Euler's constant, put

$$\mathcal{M}_{K,\Omega} = e^{\gamma}\kappa_K \prod_{\mathfrak{p}\in S}(1 - (\mathrm{N}\mathfrak{p})^{-1}). \tag{1.5}$$

We shall also use the constant

$$C_2 = \frac{\pi}{4} - \frac{\log 2}{2} + \log\left(2\left(3\log 2 - \frac{\pi}{2}\right)\right) = 0.455967\ldots. \tag{1.6}$$

Theorem 1.5. *Assume* GRH_K*.*

(i) *One has*

$$\max_{\mathfrak{a}\in\Omega(X)} L(1, \chi_{\mathfrak{a}}) \geq \mathcal{M}_{K,\Omega}(\log_2 X + \log_3 X - C_2 + o(1)).$$

(ii) *For every fixed $\eta > 0$, the proportion of $\mathfrak{a} \in \Omega(X)$ with*

$$L(1, \chi_{\mathfrak{a}}) > \mathcal{M}_{K,\Omega}(\log_2 X + \log_3 X - C_2 - \eta)$$

is at least $X^{-e^{-\eta}/2+o(1)}$.

1.1. **Earlier work.** Soundararajan introduced the resonance method for extreme values of zeta and L-functions [34, Theorems 1–2]. Its connection with GCD sums was developed by Aistleitner [1] and by Bondarenko and Seip [5, 6]. Our combinatorial construction follows the sparse prime blocks of de la Bretèche and Tenenbaum [7, Section 2.2] and retains the leading constant in their squarefree Gál estimate [7, equation (1.5), Remark after Theorem 1.1].

For quadratic Dirichlet L-functions, Darbar and Maiti [10, Theorem 1] proved, under GRH, a central maximum lower bound with coefficient 1/2 on the scale $\mathcal{V}(X)$. Dong, Wang, Zhang and Zhao [14, Theorem 1.1] obtained coefficient 1. At the moving point $1/2 + A/\log_2 X$, Dong, Li, Song and Zhao [13, Theorem 1.1] obtained $e^{-A}/(2(e-1))$. Theorem 1.1 gives

e^{-A} and counts the characters exceeding every fixed smaller threshold. At the centre over $\mathbb{Q}$, the maximum coefficient is thus the one already known. The additional conclusion is the abundance bound, which holds even in the congruence class of Corollary 1.2. Over a fixed number field, the theorem gives both assertions in the component Ω. These are lower bounds on the resonance scale, rather than asymptotic formulas for the true maximum. See [17] for conjectures on extreme values.

The point of the construction is to retain the Gál gain while keeping the fourth moment small. Proposition 5.3 provides one set with the required cardinality, prime support, shifted Gál sum and square energy. Lemma 5.5 shows that its gain survives the truncation in the approximate functional equation. The energy estimate follows from sparsity, either by the counting argument in Lemma 5.1 or by Bonami hypercontractivity, as noted in Remark 5.4. For function fields we use blocks of prescribed prime degree, where the discreteness of the norms must be taken into account.

Table 1 summarizes these comparisons. A coefficient C denotes a maximum lower bound $\exp\{(C + o(1))\mathcal{V}(X)\}$, with $\mathcal{V}_F(n)$ in place of $\mathcal{V}(X)$ over function fields. The abundance entries refer to every fixed coefficient cC, $0 < c < 1$. The number field bounds assume GRH, whereas the function field bounds are unconditional. Only the indicated maximum and abundance statements are compared.

Table 1. Central and moving-point lower bounds on the resonance scale.

Source	Family and point	Coefficient	Result compared
Darbar–Maiti [10]	Quadratic Dirichlet, $A = 0$	$1/2$	Maximum
Dong et al. [14]	Quadratic Dirichlet, $A = 0$	1	Maximum
Dong et al. [13]	Quadratic Dirichlet, $A > 0$	$\frac{e^{-A}}{2(e-1)}$	Maximum
Theorem 1.1; Corollary 1.2	Fixed number field, $A \geq 0$	e^{-A}	Maximum and $X^{1-o(1)}$ count
Darbar–Maiti [11]	Prime polynomials over $\mathbb{F}_q(t)$, $A = 0$	$1/2$	Maximum
Theorem 1.3	Principal primes over a fixed curve, $A \geq 0$	e^{-A}	Maximum and $X_n^{1-o(1)}$ count

Related results with restrictions on rational conductors appear in Fan, Hua and Xie [16], Gao [18], and Dong, Wang, Zhang and Zhao [15]. For number fields, the quadratic Hecke families of Goldmakher and Louvel [22, Definition 1 and Section 2] provide the relevant reciprocity framework. We work in an explicit Kummer component and prove the character average needed for that component directly. Moment results over imaginary quadratic fields may be found in [20, 19]. Other families of Hecke L-functions include the angular characters considered by White [36] and the class-group characters considered by Campos-Vargas [8].

Over $\mathbb{F}_q(t)$, Darbar and Maiti [11, Theorem 1] obtained coefficient $1/2$ for the central maximum in a prime-polynomial family. Theorem 1.3 gives coefficient 1, together with abundance and moving point bounds, over any fixed curve of odd characteristic. We discuss the conventions for polynomial quadratic symbols in Section 7. For large values and value distributions in function fields, see [12, 29, 28]. For central moments and limit theorems, see [4, 25, 9].

Theorems 1.4 and 1.5 use the short Euler product approach of [3] and the quadratic resonators of [10]. They give number field versions with the local factors kept explicit. Related results in the strip appear in [26, 27], and results at 1 in [30, 23, 2].

1.2. **Outline of the proof.** For a set $\mathcal{M}$ of squarefree ideals, consider

$$R_\chi = \sum_{\mathfrak{m} \in \mathcal{M}} \chi(\mathfrak{m}).$$

We estimate the first moment of $L(\sigma,\chi)$ with weight $|R_\chi|^2$ and compare it with the second moment of the resonator. After applying the approximate functional equation and averaging over the characters, square products give the main term. Given $\mathfrak{m},\mathfrak{n}\in\mathcal{M}$, the ideal $[\mathfrak{m},\mathfrak{n}]/(\mathfrak{m},\mathfrak{n})$ makes their product a square. Here parentheses and brackets denote the ideal gcd and lcm. Its contribution to the first moment contains the kernel

$$\left(\frac{\mathrm{N}(\mathfrak{m},\mathfrak{n})}{\mathrm{N}[\mathfrak{m},\mathfrak{n}]}\right)^{\sigma}.$$

The first moment lower bound is therefore governed by a Gál sum.

The fourth moment involves a different count: $E_\square(\mathcal{M})$, the number of ordered quadruples in $\mathcal{M}$ whose product is a square. For $|\mathcal{M}|=N$, we require $E_\square(\mathcal{M})\le N^{2+o(1)}$ as well as the Gál lower bound. Both estimates are obtained by taking products of sparse subsets of disjoint prime blocks. Once the fourth moment has been bounded, Cauchy–Schwarz and the individual estimate $|L(\sigma,\chi)|\le X^{o(1)}$ give the abundance assertion.

For each fixed $\varepsilon>0$, the error in the number field weighted first moment is $O(X^{3/4+\varepsilon+o(1)}N^2)$, compared with the main scale XN. We take $N=X^\beta$ with fixed $\beta<1/4$ and $\varepsilon<1/4-\beta$. The Gál constant 2 then gives $2\sqrt{\beta}$ in the maximum bound. Letting $\beta\uparrow 1/4$ gives coefficient 1. At the moving point, the primes in the blocks satisfy $\log \mathrm{N}\mathfrak{p}=(1+o(1))\log_2 N$, and $\log_2 N/\log_2 X\to 1$. The shift in each prime weight thus contributes $e^{-A}+o(1)$, uniformly for bounded A.

Sections 2–4 establish the family, the character average and the analytic estimates. The Gál construction and its truncation are proved in Section 5, and Theorem 1.1 follows in Section 6. Section 7 treats function fields. The fixed-strip and 1-line arguments occupy Sections 8 and 9.

Throughout, $O(\cdot)$, $\ll$, and $o(\cdot)$ refer to the parameter tending to infinity. Implicit constants may depend on the fixed field, the fixed component, and explicitly fixed auxiliary parameters. When a parameter varies over a compact interval, we state the required uniformity. We use $\mathbf{1}_E$ for the indicator of a condition or set E.

2. The quadratic family

We construct a family with two properties: its conductors vary with squarefree ideals, and its fixed local factors contribute no signs to the character averages. Starting from a fixed quadratic extension, we multiply its defining square class by generators of suitable principal ideals. The construction is a restricted form of that in Goldmakher–Louvel [22, Section 2], with the local conditions specified for the averages below.

Let $\mathbb{A}_K$ denote the adele ring of K. A finite-order Hecke character is a continuous character

$$\omega : K^\times\backslash\mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$$

with finite image. Its finite conductor is denoted by $\mathfrak{f}_\omega$. We say that its infinite type is trivial if its archimedean components are trivial. Its associated ideal character is

$$\chi_\omega(\mathfrak{n})=\begin{cases}\displaystyle\prod_{\mathfrak{p}^e\|\mathfrak{n}}\omega_\mathfrak{p}(\varpi_\mathfrak{p})^e, & (\mathfrak{n},\mathfrak{f}_\omega)=1,\\ 0, & (\mathfrak{n},\mathfrak{f}_\omega)>1,\end{cases}$$

where $\varpi_\mathfrak{p}$ is a uniformizer of $K_\mathfrak{p}$. At an unramified prime this value is independent of the uniformizer. The zero in this definition records ramification. A local character itself never vanishes. We use $L(s,\chi_\omega)$ and $L(s,\omega)$ for the same Hecke L-function, defined for $\Re s>1$ by the absolutely convergent expressions

$$L(s,\chi_\omega)=\sum_{\mathfrak{n}}\frac{\chi_\omega(\mathfrak{n})}{(\mathrm{N}\mathfrak{n})^s}=\prod_{\mathfrak{p}}\left(1-\frac{\chi_\omega(\mathfrak{p})}{(\mathrm{N}\mathfrak{p})^s}\right)^{-1}.$$

Here $K_\mathfrak{p}$ is the completion at $\mathfrak{p}$. We write $v_\mathfrak{p}$ for its normalized valuation. See [24, Section 3.8] for ideal and Hecke characters.

A narrow ray modulus will always include every real place. We denote its finite ideal part by $\mathfrak{q}$. If S is a finite set of finite primes, the notation $(\mathfrak{a}, S) = 1$ means that no prime in S divides $\mathfrak{a}$.

Proposition 2.1. *There are a finite set S containing the dyadic primes, a narrow ray modulus with finite part $\mathfrak{q}$ supported on S, and an ideal $\mathfrak{c}_+$ supported on S, with the following properties.*

(i) *Let Ω_+ be the squarefree integral ideals in the identity narrow ray class modulo $\mathfrak{q}$. Each $\mathfrak{a} \in \Omega_+$ determines a primitive quadratic character $\omega_\mathfrak{a}$ of trivial infinite type, with*
$$\mathfrak{f}_\mathfrak{a} = \mathfrak{c}_+\mathfrak{a}, \qquad \chi_\mathfrak{a}(\mathfrak{p}) = 0 \quad (\mathfrak{p} \in S). \tag{2.1}$$
Distinct ideals give distinct characters.

(ii) *The set Ω_+ has positive density among integral ideals. If $\mathrm{Cl}_\mathfrak{q}^+$ is the narrow ray class group, its hat denotes the character group, and μ_K is the ideal Möbius function, then*
$$\mathbf{1}_{\Omega_+}(\mathfrak{a}) = \frac{\mu_K^2(\mathfrak{a})\mathbf{1}_{(\mathfrak{a},S)=1}}{|\mathrm{Cl}_\mathfrak{q}^+|} \sum_{\xi \in \widehat{\mathrm{Cl}_\mathfrak{q}^+}} \xi(\mathfrak{a}). \tag{2.2}$$

Proof. Different totally positive generators of a principal ideal can define different quadratic extensions. We impose congruences that make every remaining unit ambiguity a square.

Let U_K^+ denote the totally positive units. The quotient U_K^+/U_K^2 is finite. For each nontrivial class choose a representative u and a distinct nondyadic prime $\mathfrak{t}_u$ at which u is not a square. Such primes exist by Chebotarev applied to $K(\sqrt{u})/K$. Let T consist of these primes and the dyadic primes.

Choose $\pi_\mathfrak{p} \in \mathfrak{p} \setminus \mathfrak{p}^2$ for each $\mathfrak{p} \in T$. The Chinese remainder theorem gives an integral d_0 with $d_0 \equiv \pi_\mathfrak{p} \pmod{\mathfrak{p}^2}$ for all $\mathfrak{p} \in T$. Choose a positive rational integer $M \in \prod_{\mathfrak{p}\in T} \mathfrak{p}^2$. Such an M exists because every nonzero integral ideal of $\mathcal{O}_K$ has nonzero intersection with $\mathbb{Z}$. For a sufficiently large positive integer k, the element $d = d_0 + kM$ is totally positive and has valuation one at every prime of T. Put
$$S = \{\mathfrak{p} : v_\mathfrak{p}(d) \text{ is odd}\}.$$
Then S contains T. In particular, every dyadic prime belongs to S. The quadratic extension $K(\sqrt{d})/K$ is ramified at every prime in S. Outside S the residue characteristic is therefore odd. Moreover, $v_\mathfrak{p}(d)$ is even, so the extension is unramified there. Write ψ_d for its quadratic character and $\mathfrak{c}_+$ for its finite conductor. Thus $\mathfrak{c}_+$ has support exactly S.

For each $\mathfrak{p} \in S$ choose $M_\mathfrak{p}$ so large that
$$1 + \mathfrak{p}^{M_\mathfrak{p}}\mathcal{O}_{K,\mathfrak{p}} \subset K_\mathfrak{p}^{\times 2}. \tag{2.3}$$
Set $\mathfrak{q} = \prod_{\mathfrak{p}\in S} \mathfrak{p}^{M_\mathfrak{p}}$, increasing the exponents if necessary so that $\mathfrak{c}_+ \mid \mathfrak{q}$. These conditions imply
$$\{u \in U_K^+ : u \equiv 1 \pmod{\mathfrak{q}}\} \subset U_K^2. \tag{2.4}$$
Indeed, a nonsquare class has a chosen prime $\mathfrak{t}_u$ where it is not a local square. Equation (2.3) excludes that class from the left side.

For $\mathfrak{a} \in \Omega_+$ choose a generator satisfying
$$(\alpha_\mathfrak{a}) = \mathfrak{a}, \qquad \alpha_\mathfrak{a} > 0 \text{ at every real place}, \qquad \alpha_\mathfrak{a} \equiv 1 \pmod{\mathfrak{q}}.$$
Define $\omega_\mathfrak{a}$ to be the quadratic character of $K(\sqrt{d\alpha_\mathfrak{a}})/K$. Any two permitted generators differ by a unit in (2.4). Hence the character is well defined. It is nontrivial because $d\alpha_\mathfrak{a}$ has odd valuation at every prime in S.

At $\mathfrak{p} \in S$, the generator $\alpha_\mathfrak{a}$ is a local square. The local character therefore agrees with ψ_d and has the same conductor exponent. Since $\mathfrak{a}$ is a ray class ideal, it is coprime to $\mathfrak{q}$, hence no prime of S divides $\mathfrak{a}$. At a prime dividing $\mathfrak{a}$, the residue characteristic is therefore odd and $d\alpha_\mathfrak{a}$ has odd valuation. The character is tamely ramified with conductor exponent one. At every remaining finite prime the residue characteristic is odd and the valuation of $d\alpha_\mathfrak{a}$ is even. The corresponding quadratic extension is unramified (possibly split). Total positivity gives trivial infinite type. This proves (2.1). The conductor recovers $\mathfrak{a}$, so distinct indices give distinct characters.

Character orthogonality in $\mathrm{Cl}_{\mathfrak{q}}^{+}$ gives (2.2). We verify its density assertion by a Dirichlet series. For a ray character ξ, write $L^S(s,\xi)$ for its Hecke L-function with the Euler factors in S removed. Then

$$\sum_{(\mathfrak{a},S)=1} \frac{\mu_K^2(\mathfrak{a})\xi(\mathfrak{a})}{(\mathrm{N}\mathfrak{a})^s} = \prod_{\mathfrak{p}\notin S}\left(1+\frac{\xi(\mathfrak{p})}{(\mathrm{N}\mathfrak{p})^s}\right) = \frac{L^S(s,\xi)}{L^S(2s,\xi^2)}. \tag{2.5}$$

The denominator is nonzero for $\Re s > 1/2$ by its absolutely convergent Euler product. Only the principal ξ contributes a pole at $s=1$. After division by $|\mathrm{Cl}_{\mathfrak{q}}^{+}|$, its residue is

$$\kappa_{\Omega_+} = \frac{\mathrm{Res}_{s=1}\,\zeta_K(s)}{|\mathrm{Cl}_{\mathfrak{q}}^{+}|\zeta_K(2)} \prod_{\mathfrak{p}\in S}\left(1+\frac{1}{\mathrm{N}\mathfrak{p}}\right)^{-1} > 0.$$

For a fixed smooth compactly supported function W on $(0,\infty)$, Mellin inversion and a shift to $\Re s = 1/2+\varepsilon$ now give

$$\sum_{\mathfrak{a}\in\Omega_+} W(\mathrm{N}\mathfrak{a}/X) = \kappa_{\Omega_+} X \int_0^\infty W(t)\,dt + O_{K,\mathfrak{q},W,\varepsilon}(X^{1/2+\varepsilon}). \tag{2.6}$$

Here the fixed Hecke L-functions have polynomial growth on vertical lines, and the Mellin transform of W decays faster than every power. No hypothesis on zeros is needed for this smoothed count. Smooth majorants and minorants of $\mathbf{1}_{[1,2]}$ can have integrals arbitrarily close to one. Applying the last estimate to them gives $\#\{\mathfrak{a}\in\Omega_+ : X < \mathrm{N}\mathfrak{a} \le 2X\} \sim \kappa_{\Omega_+}X$, proving positive density. $\square$

The use of a ramified fixed character is deliberate. For example, positive fundamental discriminants $d \equiv 5 \pmod 8$ satisfy $\chi_d(2) = -1$. A fixed congruence class alone would therefore not give nonnegative local terms in the averages below.

We henceforth fix the component $\Omega = \Omega_+$ and put $\mathfrak{c}_\Omega = \mathfrak{c}_+$. We refer to it as the positive component. The property we shall use is

$$(\mathfrak{k}, S) > 1 \implies \chi_{\mathfrak{a}}(\mathfrak{k}) = 0 \qquad (\mathfrak{a}\in\Omega). \tag{2.7}$$

The construction gives a fixed twist of the principal-ideal characters in [22, Section 2]. We need no extension to a family indexed by every squarefree ideal.

We use κ_Ω for the constant in (2.6). Under GRH_K the sharp count has the same error exponent:

$$\#\Omega(X) = \kappa_\Omega X + O_{K,\Omega,\varepsilon}(X^{1/2+\varepsilon}). \tag{2.8}$$

To see this, apply truncated Perron inversion to (2.5). For every fixed $\delta,\eta>0$, GRH bounds its numerator by $O_{K,\mathfrak{q},\delta,\eta}((1+|t|)^\eta)$ uniformly for $1/2+\delta \le \Re s \le 2$ and $|t|\ge 1$. See [24, Section 5.7]. For the principal character the pole at $s=1$ is kept as a residue, so this bound is not asserted near that pole. On the new vertical line $\Re s = 1/2+\delta < 1$, the compact part $|t|\le 1$ is bounded separately. The inverse denominator is bounded throughout the strip by absolute convergence. The coefficient of n^{-s} is bounded in modulus by the number of ideals of norm n, which is at most $d_{d_K}(n) \ll_{K,\eta} n^\eta$. Replace Y by a half-integer with the same integer part, and take the Perron height $T = Y^2$ and the initial line $\Re s = 1 + 1/\log Y$. This replacement preserves the summatory function, since ideal norms are integers, and changes the residue main term by only $O_{K,\mathfrak{q}}(1)$. The truncation error is $O_{K,\mathfrak{q},\eta}(Y^{1+\eta}/T)$, with a logarithm absorbed into Y^η. Shifting to $\Re s = 1/2+\delta$ contributes $O(Y^{1/2+\delta}T^\eta \log T + Y^{1+\eta}T^{-1+\eta})$. For $0 < \varepsilon < 1/2$, take $\delta = \varepsilon/4$ and $\eta = \varepsilon/16$. Then $\delta + 2\eta < \varepsilon$ and all these errors are $O_{K,\mathfrak{q},\varepsilon}(Y^{1/2+\varepsilon})$. Larger ε follow from this range. The principal pole gives the residue already computed, and subtracting the counts at $2X$ and X proves (2.8).

For $(\mathfrak{n}, S) = 1$ put

$$h_K(\mathfrak{n}) = \prod_{\mathfrak{p}|\mathfrak{n}} \frac{\mathrm{N}\mathfrak{p}}{\mathrm{N}\mathfrak{p}+1}. \tag{2.9}$$

This factor is the density correction for requiring a squarefree index to be coprime to $\mathfrak{n}$.

3. Averages of characters

Square arguments contribute the main term in the character average. For nonsquare arguments, reciprocity gives a nonprincipal character in the family index, to which we can apply GRH. Since the argument ideal need not be principal, we construct this reciprocal character on a ray class group. Its conductor bound supplies the required uniformity.

Fix an integer $J_0 > 2d_K + 10$. For a smooth function w supported on $[X, 2X]$, define

$$\|w\|_{*,X} = \sum_{j=0}^{J_0} X^j \|w^{(j)}\|_\infty.$$

This norm allows the weight to vary with terms in an approximate functional equation.

Proposition 3.1. *Assume* GRH_K. *Let w be smooth and supported on $[X, 2X]$. Write $\mathfrak{n} = \mathfrak{n}_0\mathfrak{n}_1^2$, where $\mathfrak{n}_0$ is squarefree, and assume $(\mathfrak{n}, S) = 1$. For every fixed $0 < \varepsilon < 1/2$,*

$$\sum_{\mathfrak{a}\in\Omega} w(\mathrm{N}\mathfrak{a})\chi_{\mathfrak{a}}(\mathfrak{n}) = \kappa_\Omega h_K(\mathfrak{n})\mathbf{1}_{\mathfrak{n}=\square} \int_0^\infty w(t)\,dt + O_{K,\Omega,\varepsilon}\left(\|w\|_{*,X} X^{1/2+\varepsilon} F_\varepsilon(\mathfrak{n}_0) G_\varepsilon(\mathfrak{n}_1)\right), \tag{3.1}$$

where the conductor and coprimality factors may be taken as

(i) $F_\varepsilon(\mathfrak{n}_0) = \exp\{C_{K,\varepsilon}(\log(2 + \mathrm{N}\mathfrak{n}_0))^{1-\varepsilon}\}$;

(ii) $G_\varepsilon(\mathfrak{n}_1) = \prod_{\mathfrak{p}|\mathfrak{n}_1} \left(1 + (\mathrm{N}\mathfrak{p})^{-1/2-\varepsilon/2}\right)$.

Proof. Let H be the kernel of the natural homomorphism $\mathrm{Cl}^+_{\mathfrak{q}\mathfrak{n}_0} \to \mathrm{Cl}^+_{\mathfrak{q}}$. Each class in H is represented by a principal ideal (α), coprime to $\mathfrak{q}\mathfrak{n}_0$, with α totally positive and $\alpha \equiv 1 \pmod{\mathfrak{q}}$. Set

$$\rho_{\mathfrak{n}_0}((\alpha)) = \prod_{\mathfrak{p}|\mathfrak{n}_0} \left(\frac{\alpha}{\mathfrak{p}}\right), \tag{3.2}$$

where the factors are quadratic residue symbols in the residue fields. The primes involved are nondyadic. Changing the generator multiplies it by a square, by (2.4). Changing the representative within its $\mathfrak{q}\mathfrak{n}_0$-ray class multiplies it by an element congruent to one at every prime of $\mathfrak{n}_0$. Thus (3.2) is a well-defined character of H.

A character of a subgroup of a finite abelian group extends to the whole group. Choose such an extension to $\mathrm{Cl}^+_{\mathfrak{q}\mathfrak{n}_0}$ and retain the notation $\rho_{\mathfrak{n}_0}$. For $\mathfrak{n}_0 = (1)$ choose the principal extension. The extension has finite order, but need not have order two. This is why we assume GRH for all finite-order Hecke characters.

Its finite conductor divides $\mathfrak{q}\mathfrak{n}_0$. The extension to the narrow ray class group may carry an archimedean sign type, but there are only finitely many such types (depending on K). They are absorbed in the constants below and do not affect the finite-conductor divisibility. We also need ramification at every $\mathfrak{p} \mid \mathfrak{n}_0$. Choose a totally positive α congruent to one modulo $\mathfrak{q}\prod_{\mathfrak{l}|\mathfrak{n}_0,\,\mathfrak{l}\neq\mathfrak{p}} \mathfrak{l}$ and to a nonzero nonsquare modulo $\mathfrak{p}$. The Chinese remainder theorem, followed by addition of a sufficiently large positive rational integer in the modulus, gives such an α. The class of (α) becomes trivial when $\mathfrak{p}$ is removed from the ray modulus, whereas $\rho_{\mathfrak{n}_0}((\alpha)) = -1$. Every extension is therefore ramified at $\mathfrak{p}$, and

$$\mathfrak{n}_0 \mid \mathfrak{f}_{\rho_{\mathfrak{n}_0}} \mid \mathfrak{q}\mathfrak{n}_0.$$

Let $\psi_{\alpha_{\mathfrak{a}}}$ denote the character of $K(\sqrt{\alpha_{\mathfrak{a}}})/K$. For $(\mathfrak{a},\mathfrak{n})=1$ we have $\omega_{\mathfrak{a}}=\psi_d\psi_{\alpha_{\mathfrak{a}}}$ and

$$\chi_{\mathfrak{a}}(\mathfrak{n})=\psi_d(\mathfrak{n})\rho_{\mathfrak{n}_0}(\mathfrak{a}). \tag{3.3}$$

Here $\psi_d(\mathfrak{n})$ denotes its ideal value. Since $(\mathfrak{a},\mathfrak{n})=1$, every prime $\mathfrak{p}\mid\mathfrak{n}_0$ is unramified in $K(\sqrt{\alpha_{\mathfrak{a}}})/K$. The unramified quadratic local Artin character evaluates on a uniformizer as the quadratic residue symbol, so

$$\psi_{\alpha_{\mathfrak{a}}}(\mathfrak{p})=\left(\frac{\alpha_{\mathfrak{a}}}{\mathfrak{p}}\right).$$

This identity is independent of whether one uses arithmetic or geometric Frobenius, since the value is in $\{\pm1\}$ and hence equals its inverse. The square part $\mathfrak{n}_1^2$ contributes one. Hence

$$\psi_{\alpha_{\mathfrak{a}}}(\mathfrak{n})=\prod_{\mathfrak{p}\mid\mathfrak{n}_0}\left(\frac{\alpha_{\mathfrak{a}}}{\mathfrak{p}}\right)=\rho_{\mathfrak{n}_0}(\mathfrak{a}),$$

where the last equality follows because the class of $\mathfrak{a}$ in $\mathrm{Cl}^+_{\mathfrak{q}\mathfrak{n}_0}$ lies in H (it is principal with a totally positive generator congruent to 1 modulo $\mathfrak{q}$), and the chosen extension $\rho_{\mathfrak{n}_0}$ restricts to (3.2) on H. This proves (3.3). If $(\mathfrak{a},\mathfrak{n})>1$, the character value is zero by (2.1). In the nonsquare case the factor $\psi_d(\mathfrak{n})$ is independent of $\mathfrak{a}$ and has absolute value one, so it may be suppressed in the error estimate below. In the square case it equals one.

Insert (2.2). For each ray character ξ put $\rho=\xi\rho_{\mathfrak{n}_0}$ and let T consist of S and the primes dividing $\mathfrak{n}$. The required Dirichlet series is

$$D_{\xi,\mathfrak{n}}(s)=\sum_{(\mathfrak{a},T)=1}\frac{\mu_K^2(\mathfrak{a})\rho(\mathfrak{a})}{(\mathrm{N}\mathfrak{a})^s}=\frac{L^T(s,\rho)}{L^T(2s,\rho^2)}.$$

Here $L^T(s,\rho)$ denotes the primitive Hecke L-function inducing ρ, with all Euler factors in T omitted. The same convention is used for ρ^2, whose primitive conductor can be smaller. For $\Re s>1/2$, the Euler product for $1/L^T(2s,\rho^2)$ converges absolutely and locally uniformly. Thus it is holomorphic there, and the only possible pole of $D_{\xi,\mathfrak{n}}$ in this half-plane comes from a principal numerator at $s=1$. In particular, no zero-free estimate on the line $\Re(2s)=1$ is needed. If ρ^2 is principal, then

$$L^T(2s,\rho^2)=\zeta_K(2s)\prod_{\mathfrak{p}\in T}(1-(\mathrm{N}\mathfrak{p})^{-2s}),$$

$$\frac{1}{L^T(2s,\rho^2)}=O_{K,T}(s-1/2)$$

near $s=1/2$. The pole of this denominator therefore gives a zero of its reciprocal, not an additional residue. The contour below stays strictly to the right of this point. If $\mathfrak{n}_0\neq(1)$, then ρ is ramified at every prime dividing $\mathfrak{n}_0$: multiplication by ξ, whose conductor is supported on S, cannot remove that ramification. Thus ρ is nonprincipal. If $\mathfrak{n}_0=(1)$, then $\rho=\xi$, so only the principal ray character can contribute a pole at one.

We estimate the same shifted integral in both cases. For a nonprincipal primitive character inducing ρ, the standard Littlewood bound under GRH in terms of the analytic conductor implies, at $s=1/2+\varepsilon+it$,

$$\log|L(s,\rho)|\ll_{K,\varepsilon}\frac{(\log C(\rho,t))^{1-2\varepsilon}}{\log\log(3+C(\rho,t))},$$

with the harmless convention that the denominator is bounded below by a positive constant. Since $C(\rho,t)\ll_{K,\Omega}\mathrm{N}\mathfrak{f}_\rho(3+|t|)^{d_K}$ and $\mathfrak{f}_\rho\mid\mathfrak{q}\mathfrak{n}_0$, enlarging $C_{K,\varepsilon}$ gives the coarser bound needed here,

$$|L(1/2+\varepsilon+it,\rho)|\ll_{K,\Omega,\varepsilon}(1+|t|)F_\varepsilon(\mathfrak{n}_0).$$

Indeed the conductor part is dominated by $(\log(2+\mathrm{N}\mathfrak{n}_0))^{1-\varepsilon}$, while the height contribution is $o_\varepsilon(\log(2+|t|))$ as $|t|\to\infty$. It is therefore at most $\log(2+|t|)+O_{K,\varepsilon}(1)$, which justifies the displayed linear factor rather than an unspecified power of the height. If ρ is principal, its primitive function is the fixed ζ_K: the same bound on this shifted line follows from the corresponding Littlewood estimate at large height. The remaining compact part of the line avoids its pole and is bounded with a constant depending only on K, ε. Its pole at $s = 1$ is handled separately by a residue. See [24, Section 5.7]. The local parameters of these finite-order Hecke functions have modulus at most one, so the required Ramanujan bound is automatic.

To keep the dependence on $\mathfrak{n}_1$ explicit, first omit only the primes in S and those dividing $\mathfrak{n}_0$. The inverse denominator is uniformly bounded by the convergent product $\prod_{\mathfrak{p}}(1+(\mathrm{N}\mathfrak{p})^{-1-2\varepsilon})$. In the numerator, primes dividing $\mathfrak{n}_0$ are ramified, and the finitely many primes in S cost a fixed constant. Removing an additional prime $\mathfrak{p} \mid \mathfrak{n}_1$ not already in $S \cup \{\mathfrak{p} : \mathfrak{p} \mid \mathfrak{n}_0\}$ multiplies this squarefree Euler product by $(1 + \rho(\mathfrak{p})(\mathrm{N}\mathfrak{p})^{-s})^{-1}$. For all sufficiently large $P = \mathrm{N}\mathfrak{p}$, depending only on ε,

$$|1 + \rho(\mathfrak{p})P^{-s}|^{-1} \leq \frac{1}{1 - P^{-1/2-\varepsilon}} \leq 1 + P^{-1/2-\varepsilon/2}.$$

Only finitely many smaller primes remain. Their product is bounded by a constant depending on K and ε. Consequently

$$|D_{\xi,\mathfrak{n}}(1/2 + \varepsilon + it)| \ll_{K,\Omega,\varepsilon} (1 + |t|)F_\varepsilon(\mathfrak{n}_0)G_\varepsilon(\mathfrak{n}_1).$$

Let $\widehat{w}(s) = \int_0^\infty w(u)u^{s-1}\,du$. Integration by parts J_0 times, after the substitution $u = Xv$, gives

$$|\widehat{w}(\sigma + it)| \ll_{J_0} \|w\|_{*,X} X^\sigma (1 + |t|)^{-J_0} \qquad (1/2 \leq \sigma \leq 2).$$

For each fixed $\mathfrak{n}$ and ρ, the standard vertical-strip growth estimate for the numerator, and the absolutely convergent inverse denominator, give $D_{\xi,\mathfrak{n}}(v+it) = O_{K,\mathfrak{n},\varepsilon}((1+|t|)^{d_K+2})$ uniformly for $1/2 + \varepsilon \leq v \leq 2$ and $|t| \geq 1$. The finitely many removed Euler factors are bounded throughout this strip. Since $J_0 > 2d_K + 10$, the horizontal integrals tend to zero. Mellin inversion and a contour shift from $\Re s = 2$ to $\Re s = 1/2+\varepsilon$ therefore give the stated error, using the uniform bound on the new vertical line. The constants used only to let the horizontal integrals vanish need not be uniform in $\mathfrak{n}$. When $\mathfrak{n}$ is a square, the principal residue from (2.6) is multiplied by

$$\prod_{\mathfrak{p}\mid\mathfrak{n}} \left(1 + \frac{1}{\mathrm{N}\mathfrak{p}}\right)^{-1} = h_K(\mathfrak{n}).$$

In this case $\psi_d(\mathfrak{n}) = 1$. When $\mathfrak{n}$ is not a square, there is no pole. These are exactly the two main terms in (3.1). □

The error factors permit very large ideals when their prime divisors are small. The following form is convenient for the resonators below.

Lemma 3.2. *Fix $B_0 > 0$ and $0 < \varepsilon < 1/2$. Suppose $\mathfrak{n} = \mathfrak{u}\mathfrak{v}$ satisfies*

(i) $\mathrm{N}\mathfrak{u} \leq X^{B_0}$;

(ii) *every prime divisor of $\mathfrak{v}$ has norm at most $z \leq (\log X)^{1+\varepsilon/4}$.*

Writing $\mathfrak{n} = \mathfrak{n}_0\mathfrak{n}_1^2$ with $\mathfrak{n}_0$ squarefree, we have uniformly

$$\log\{F_\varepsilon(\mathfrak{n}_0)G_\varepsilon(\mathfrak{n}_1)\} \ll_{K,B_0,\varepsilon} (\log X)^{1-\varepsilon/2} = o(\log X).$$

In particular the product is $X^{o(1)}$, with one bound for all such $\mathfrak{u}, \mathfrak{v}$, irrespective of the prime multiplicities in $\mathfrak{v}$.

Proof. The prime ideal theorem gives $\sum_{\mathrm{N}\mathfrak{p}\leq z} \log \mathrm{N}\mathfrak{p} \ll_K z$. Hence

$$\log \mathrm{N}\mathfrak{n}_0 \leq B_0 \log X + O_K(z),$$

$$\log F_\varepsilon(\mathfrak{n}_0) \ll_{K,B_0,\varepsilon} (\log X + z)^{1-\varepsilon} \ll_{K,B_0,\varepsilon} (\log X)^{1-\varepsilon/2}.$$

The last inequality uses $(1 + \varepsilon/4)(1 - \varepsilon) \leq 1 - \varepsilon/2$.

Put $a = 1/2 + \varepsilon/2$. The primes coming from $\mathfrak{v}$ contribute at most

$$\sum_{\mathrm{N}\mathfrak{p} \le z} (\mathrm{N}\mathfrak{p})^{-a} \ll_{K,\varepsilon} \frac{z^{1-a}}{\log(2+z)} = o(\log X)$$

to $\log G_\varepsilon(\mathfrak{n}_1)$. For the primes dividing $\mathfrak{u}$, split at $y = \log X$. The primes of norm at most y contribute $O_{K,\varepsilon}(y^{1-a}/\log y)$ by partial summation. There are at most $B_0 \log X/\log y$ prime divisors of norm greater than y, since their product divides $\mathfrak{u}$. Their contribution is at most

$$\frac{B_0 \log X}{\log y} y^{-a} = B_0 \frac{(\log X)^{1-a}}{\log_2 X} = o(\log X).$$

Since $(1+\varepsilon/4)(1-a) < 1-\varepsilon/2$, these estimates also bound $\log G_\varepsilon(\mathfrak{n}_1)$ by $O_{K,B_0,\varepsilon}((\log X)^{1-\varepsilon/2})$. Combining the bounds proves the asserted uniform estimate. $\square$

4. The approximate functional equation and bounds under GRH

The resonance argument uses two analytic facts. We need an approximate functional equation whose weights are nonnegative. We also need a uniform upper bound to turn a weighted lower bound into a count of characters. We prove the central and shifted versions together.

Let r_1 and r_2 be the numbers of real and complex places of K. For $\mathfrak{a} \in \Omega$, put

$$\mathcal{Q}_{\mathfrak{a}} = |D_K|\mathrm{N}\mathfrak{f}_{\mathfrak{a}}, \qquad \gamma_K(s) = \pi^{-r_1 s/2}(2\pi)^{-r_2 s}\Gamma(s/2)^{r_1}\Gamma(s)^{r_2}.$$

Here D_K is the discriminant of K. The omitted constant in each complex gamma factor cancels in every ratio below. Since the characters have trivial infinite type, their completed functions are

$$\Lambda(s,\chi_{\mathfrak{a}}) = \mathcal{Q}_{\mathfrak{a}}^{s/2}\gamma_K(s)L(s,\chi_{\mathfrak{a}}).$$

We record explicitly why this normalization has root number $+1$. Put $L_{\mathfrak{a}} = K(\sqrt{d\alpha_{\mathfrak{a}}})$. The conductor–discriminant formula for the nontrivial quadratic Artin character gives

$$|D_{L_{\mathfrak{a}}}| = |D_K|^2 \mathrm{N}\mathfrak{f}_{\mathfrak{a}}.$$

Moreover $d\alpha_{\mathfrak{a}}$ is totally positive, so every real place of K splits in $L_{\mathfrak{a}}$. Consequently the quotient of the archimedean factors of $\zeta_{L_{\mathfrak{a}}}$ and ζ_K is precisely $\gamma_K(s)$ (up to the harmless s-independent constant allowed by the standard choice of complex gamma factor), while the quotient of the conductor factors is $\mathcal{Q}_{\mathfrak{a}}^{s/2}$. Artin factorization therefore identifies $\Lambda(s,\chi_{\mathfrak{a}})$, up to such an s-independent positive constant, with the quotient of the completed Dedekind zeta functions of $L_{\mathfrak{a}}$ and K. Since both Dedekind zeta functions satisfy functional equations with sign $+1$, the constant cancels and

$$\Lambda(s,\chi_{\mathfrak{a}}) = \Lambda(1-s,\chi_{\mathfrak{a}}).$$

The character is nontrivial, and the completed primitive Hecke L-function is entire. Thus no pole is crossed in the contour shift below except the pole of $1/z$ at $z = 0$. These are the standard functional-equation facts for quadratic Hecke characters. See [24, Chapter 5].

Lemma 4.1. *Fix $0 < \delta_0 < 1/4$. For $|\delta| \le \delta_0$, define $s_\delta = 1/2 + \delta$ and*

$$V_{K,\delta}(x) = \frac{1}{2\pi i}\int_{(c)} \frac{\gamma_K(s_\delta + z)}{\gamma_K(s_\delta)} x^{-z}\frac{dz}{z}, \qquad c > 0, \tag{4.1}$$

where $\int_{(c)}$ means integration upward along $\Re z = c$. The value is independent of the particular $c > 0$ by contour shifting inside the half-plane before the first gamma pole. Then the following assertions hold.

(i) *For* $0 \le \delta \le \delta_0$,

$$L(s_\delta, \chi_{\mathfrak{a}}) = \sum_{\mathfrak{k}} \frac{\chi_{\mathfrak{a}}(\mathfrak{k})}{(\mathrm{N}\mathfrak{k})^{s_\delta}} V_{K,\delta}\left(\frac{\mathrm{N}\mathfrak{k}}{\sqrt{\mathcal{Q}_{\mathfrak{a}}}}\right) + Y_{\mathfrak{a},\delta} \sum_{\mathfrak{k}} \frac{\chi_{\mathfrak{a}}(\mathfrak{k})}{(\mathrm{N}\mathfrak{k})^{1-s_\delta}} V_{K,-\delta}\left(\frac{\mathrm{N}\mathfrak{k}}{\sqrt{\mathcal{Q}_{\mathfrak{a}}}}\right), \tag{4.2}$$

where

$$Y_{\mathfrak{a},\delta} = \mathcal{Q}_{\mathfrak{a}}^{-\delta} \frac{\gamma_K(1/2-\delta)}{\gamma_K(1/2+\delta)} > 0. \tag{4.3}$$

(ii) *The kernels take values in* $[0,1]$. *For* $0 < \rho < 1/2 - \delta_0$, *uniformly in* $|\delta| \le \delta_0$,

$$V_{K,\delta}(x) = 1 + O_{K,\delta_0,\rho}(x^\rho), \qquad 0 < x \le 1. \tag{4.4}$$

For every integer $j \ge 0$ *and every* $B \ge 0$,

$$\left|(x\tfrac{d}{dx})^j V_{K,\delta}(x)\right| \ll_{K,\delta_0,j,B} (1+x)^{-B}. \tag{4.5}$$

In particular, if $V_K = V_{K,0}$, *then*

$$L(1/2, \chi_{\mathfrak{a}}) = 2 \sum_{\mathfrak{k}} \frac{\chi_{\mathfrak{a}}(\mathfrak{k})}{\sqrt{\mathrm{N}\mathfrak{k}}} V_K\left(\frac{\mathrm{N}\mathfrak{k}}{\sqrt{\mathcal{Q}_{\mathfrak{a}}}}\right). \tag{4.6}$$

Proof. The contour argument is the usual approximate functional equation [24, Section 5.2]. The central rational-field instance appears in [33, Lemmas 2.1–2.2, pp. 455–456]. We give the calculation because the positivity and uniformity of the shifted weights will be used later.

Choose $c > 1/2 + \delta_0$ and set

$$I_\delta = \frac{1}{2\pi i} \int_{(c)} \frac{\Lambda(s_\delta + z, \chi_{\mathfrak{a}})}{\mathcal{Q}_{\mathfrak{a}}^{s_\delta/2} \gamma_K(s_\delta)} \frac{dz}{z}.$$

Move the line to $\Re z = -c$. The gamma factors decay exponentially, whereas the Hecke function has polynomial growth in a fixed vertical strip. Thus the horizontal integrals vanish. Entireness leaves only the residue at $z = 0$. In the remaining integral substitute $z = -w$ and use the functional equation. The reversed orientation gives

$$I_\delta = L(s_\delta, \chi_{\mathfrak{a}}) - Y_{\mathfrak{a},\delta} I_{-\delta}. \tag{4.7}$$

Both Dirichlet series converge absolutely on the right-hand lines. The dual Dirichlet series normally contains the contragredient character. Here $\chi_{\mathfrak{a}}$ is real quadratic, so $\overline{\chi_{\mathfrak{a}}} = \chi_{\mathfrak{a}}$. Expanding the two integrals therefore gives (4.2).

To see why the weights are positive, let $s \in [1/2 - \delta_0, 1/2 + \delta_0]$. Take independent random variables U_j and Z_j with respective densities $t^{s/2-1}e^{-t}/\Gamma(s/2)$ and $t^{s-1}e^{-t}/\Gamma(s)$ on $t > 0$. Put

$$T_s = \prod_{j=1}^{r_1} (U_j/\pi)^{1/2} \prod_{j=1}^{r_2} (Z_j/(2\pi)).$$

The gamma integral gives $\mathbf{E}T_s^z = \gamma_K(s+z)/\gamma_K(s)$. Mellin inversion, or Perron's formula applied inside this expectation, gives

$$V_{K,\delta}(x) = \mathbf{P}(T_{s_\delta} > x). \tag{4.8}$$

The distributions are continuous. Consequently the identity holds for every $x > 0$, and $0 \le V_{K,\delta}(x) \le 1$. Moreover,

$$1 - V_{K,\delta}(x) \le x^\rho \mathbf{E} T_{s_\delta}^{-\rho} \ll_{K,\delta_0,\rho} x^\rho,$$

since $\rho < 1/2 - \delta_0$ keeps all gamma arguments positive.

For $x \ge 1$, differentiate the Mellin integral and move its line to $\Re z = B + 1$. Stirling's formula proves (4.5). For $0 < x \le 1$ and $j \ge 1$, the factor $(-z)^j$ removes the pole at zero. Move the line to $\Re z = -\rho$. No gamma pole is crossed. The result is $O(x^\rho)$. For $j = 0$, the probability bound suffices. All constants are uniform on the stated interval of δ. □

For the abundance argument, we need an individual bound with exponent $o(\log X)$. The following estimate suffices.

Lemma 4.2. *Assume* GRH_K*. For every fixed* $A_0 \geq 0$*, uniformly in* $\mathfrak{a} \in \Omega(X)$ *and* $0 \leq \delta \leq A_0/\log_2 X$*,*

$$|L(1/2+\delta,\chi_{\mathfrak{a}})| \leq \exp\left(C_{K,\Omega}\frac{\log X}{\log_2 X}\right) = X^{o(1)}.$$

Proof. We use the logarithmic majorant of Soundararajan [35, Proposition, p. 984, and Section 2]. The following calculation verifies it for our Hecke functions and shifts. Write $L(s) = L(s,\chi_{\mathfrak{a}})$ and $\mathcal{Q} = \mathcal{Q}_{\mathfrak{a}}$. Under GRH the nontrivial zeros are $\rho = 1/2 + i\gamma$. For real $u > 1/2$, let

$$F(u) = \sum_{\rho} \frac{u-1/2}{(u-1/2)^2+\gamma^2} \geq 0,$$

where zeros are counted with multiplicity. Hadamard factorization gives

$$-\Re\frac{L'}{L}(u) = \tfrac{1}{2}\log\mathcal{Q} + O_K(1) - F(u) \qquad (1/2 < u \leq 1). \tag{4.9}$$

Here the gamma logarithmic derivatives are bounded on the indicated real interval, and the nonprincipal character contributes no pole.

Let $x \geq e^4$, $\Delta = 1/\log x$, and $\sigma_0 = \sigma + \Delta$, where $1/2 \leq \sigma \leq 3/4$. For $a_0 = \sigma_0 - 1/2 \geq \Delta$, each zero contributes

$$\int_{\sigma}^{\sigma_0} \frac{u-1/2}{(u-1/2)^2+\gamma^2}\,du = \frac{1}{2}\log\frac{a_0^2+\gamma^2}{(a_0-\Delta)^2+\gamma^2} \geq \frac{\Delta}{2}\frac{a_0}{a_0^2+\gamma^2}.$$

Indeed, $a_0^2 - (a_0-\Delta)^2 = \Delta(2a_0-\Delta) \geq \Delta a_0$, and $\log(1+y) \geq y/(1+y)$. At a central zero the integral is infinite and the eventual upper bound for $|L(1/2)|$ is immediate. Otherwise all endpoint assertions follow by a limit from the right. Integration of (4.9) therefore yields

$$\log|L(\sigma)| - \log|L(\sigma_0)| \leq \frac{\Delta}{2}\log\mathcal{Q} - \frac{\Delta}{2}F(\sigma_0) + O_K(\Delta). \tag{4.10}$$

Let $\Lambda_K(\mathfrak{n})$ be the ideal von Mangoldt function: it is $\log \mathrm{N}\mathfrak{p}$ if $\mathfrak{n} = \mathfrak{p}^j$ with $j \geq 1$, and zero otherwise. For real $s > 1/2$, apply Perron's formula to $-L'/L$ with kernel x^w/w^2, initially on $\Re w > \max\{0, 1-s\}$. The residue at $w = 0$ is $-(L'/L)(s)\log x - (L'/L)'(s)$. A zero α of multiplicity m_α contributes $-m_\alpha x^{\alpha-s}/(\alpha-s)^2$. Shifting left along contours avoiding the zeros, then dividing by $\log x$ and rearranging, gives

$$\begin{aligned}-\frac{L'}{L}(s) = &\sum_{\mathrm{N}\mathfrak{n}\leq x} \frac{\Lambda_K(\mathfrak{n})\chi_{\mathfrak{a}}(\mathfrak{n})}{(\mathrm{N}\mathfrak{n})^s}\frac{\log(x/\mathrm{N}\mathfrak{n})}{\log x} + \frac{(L'/L)'(s)}{\log x} \\ &+ \frac{1}{\log x}\sum_{\rho}\frac{x^{\rho-s}}{(\rho-s)^2} + \mathcal{E}_K(s,x).\end{aligned} \tag{4.11}$$

There is no residue from a pole of L, since the primitive character is nonprincipal. The gamma factors specified above give the trivial zero at $-m$, $m \geq 0$, multiplicity $b_m = r_2 + r_1\mathbf{1}_{2|m}$. This includes the zero at 0: the functional equation and $L(1) \neq 0$ show that the completed function is nonzero there, so cancellation of the gamma pole is exact. Consequently

$$\mathcal{E}_K(u,x) = \frac{1}{\log x}\sum_{m\geq 0} b_m \frac{x^{-u-m}}{(u+m)^2},$$

$$\int_{\sigma_0}^{\infty} |\mathcal{E}_K(u,x)|\,du \ll_K \frac{x^{-\sigma_0}}{(\log x)^2}.$$

Here $\sigma_0 \geq 1/2$ and $x \geq e^4$, so the bound is uniform even as $\sigma \downarrow 1/2$. The residue formula follows first with finite contours. The usual logarithmic-derivative bounds away from zeros justify their limit, as in [35, Section 2].

Take real parts and integrate from σ_0 to infinity. The derivative term integrates to $-\Re(L'/L)(\sigma_0)/\log x$, since $L'/L(u) \to 0$ as $u \to \infty$. The nontrivial zero sum can be integrated termwise: $\sum_\rho |\sigma_0-\rho|^{-2} < \infty$ for $\sigma_0 > 1/2$, and $|u-\rho| \ge |\sigma_0-\rho|$ for $u \ge \sigma_0$. Its absolute contribution is at most

$$\frac{1}{\log x}\sum_\rho \int_{\sigma_0}^{\infty} \frac{x^{1/2-u}}{|\sigma_0-\rho|^2}\,du = \frac{x^{1/2-\sigma_0}}{(\sigma_0-1/2)(\log x)^2}F(\sigma_0).$$

Using (4.9) only at $\sigma_0 \le 1$ (not over the entire unbounded interval of integration), we obtain

$$\begin{aligned}\log|L(\sigma_0)| \le \Re \sum_{\mathrm{N}\mathfrak{n}\le x} \frac{\Lambda_K(\mathfrak{n})\chi_{\mathfrak{a}}(\mathfrak{n})}{(\mathrm{N}\mathfrak{n})^{\sigma_0}\log \mathrm{N}\mathfrak{n}}\frac{\log(x/\mathrm{N}\mathfrak{n})}{\log x} + \frac{\log \mathcal{Q}}{2\log x}\\ + \left\{\frac{x^{1/2-\sigma_0}}{(\sigma_0-1/2)(\log x)^2} - \frac{1}{\log x}\right\}F(\sigma_0) + O_K(1).\end{aligned} \tag{4.12}$$

Add (4.10). Writing $y = (\sigma_0 - 1/2)\log x \ge 1$, the positive part of the resulting $F(\sigma_0)$-coefficient is $e^{-y}/(y\log x) \le e^{-1}/\log x$, while the two negative contributions total $-3/(2\log x)$. Hence the coefficient of $F(\sigma_0)$ is at most $(e^{-1}-3/2)/\log x < 0$. This is exactly the $\lambda = 1$ instance of the sign mechanism in Soundararajan's majorant, now with a shift starting at any $\sigma \in [1/2, 3/4]$. The conductor contributions are $\Delta \log \mathcal{Q}/2 + \log\mathcal{Q}/(2\log x) = \log \mathcal{Q}/\log x$. Removing the nonpositive zero term gives

$$\log|L(\sigma)| \le \Re \sum_{\mathrm{N}\mathfrak{n}\le x} \frac{\Lambda_K(\mathfrak{n})\chi_{\mathfrak{a}}(\mathfrak{n})}{(\mathrm{N}\mathfrak{n})^{\sigma+1/\log x}\log \mathrm{N}\mathfrak{n}}\frac{\log(x/\mathrm{N}\mathfrak{n})}{\log x} + \frac{\log\mathcal{Q}}{\log x} + O_K(1). \tag{4.13}$$

At a central zero the claimed upper bound is immediate. Elsewhere the endpoint $\sigma = 1/2$ follows by a limit.

Take $x = (\log X)^2$. The prime ideal theorem and partial summation give

$$\sum_{(\mathrm{N}\mathfrak{p})^j \le x} \frac{1}{j(\mathrm{N}\mathfrak{p})^{j/2}} \ll_K \frac{\sqrt{x}}{\log x} + \log_2 x \ll_K \frac{\log X}{\log_2 X}.$$

Since $\mathcal{Q} \asymp_{K,\Omega} X$, the conductor term has the same bound. For every fixed A_0, once $\log_2 X \ge 4A_0$ the required interval $\sigma = 1/2 + A/\log_2 X$, $0 \le A \le A_0$, lies in $[1/2, 3/4]$. Every estimate above is uniform on that entire interval. Only the threshold for X depends on A_0, while the displayed constant depends on K, Ω. Exponentiation finishes the proof. □

For a fixed point to the right of $1/2$, the ordinary short Euler product is more convenient than the approximate functional equation.

Lemma 4.3. *Assume* GRH_K *and fix* $1/2 < \sigma \le 1$. *For* $y = (\log X)^B$ *with* B *sufficiently large in terms of* σ, *the following estimates hold uniformly on* $\Omega(X)$.

(i) *If* $\sigma < 1$, *then*

$$\log L(\sigma, \chi_{\mathfrak{a}}) = \sum_{\mathrm{N}\mathfrak{p}\le y} \frac{\chi_{\mathfrak{a}}(\mathfrak{p})}{(\mathrm{N}\mathfrak{p})^\sigma} + O_{K,\sigma}(1). \tag{4.14}$$

(ii) *At* $\sigma = 1$,

$$L(1,\chi_{\mathfrak{a}}) = \prod_{\mathrm{N}\mathfrak{p}\le y}\left(1 - \frac{\chi_{\mathfrak{a}}(\mathfrak{p})}{\mathrm{N}\mathfrak{p}}\right)^{-1}\{1 + O_{K,\Omega}((\log X)^{-10})\}. \tag{4.15}$$

Proof. GRH and the explicit formula for a nonprincipal finite-order Hecke character give

$$\Psi_\chi(t) := \sum_{\mathrm{N}\mathfrak{n}\le t} \Lambda_K(\mathfrak{n})\chi(\mathfrak{n}) \ll_K t^{1/2}\log^2\big(\mathcal{Q}(t+2)^{d_K}\big);$$

see [24, Chapter 5]. Partial summation therefore makes the series for $\log L(\sigma,\chi)$ converge for $\sigma > 1/2$, and bounds its tail by

$$\left|\int_y^\infty \frac{t^{-\sigma}}{\log t}\, d\Psi_\chi(t)\right| \ll_{K,\sigma} y^{1/2-\sigma}\frac{\log^2(\mathcal{Q}(y+2)^{d_K})}{\log y}.$$

The identity with $\log L$ first holds for $\sigma > 1$ and extends throughout $\sigma > 1/2$ by analytic continuation. Its branch is real on the real axis: L is positive for $\sigma > 1$ and has no zeros for $\sigma > 1/2$ under GRH. Choosing B large makes this tail $O((\log X)^{-10})$. The terms with prime-power exponent at least two have bounded total absolute value when $\sigma > 1/2$, proving (4.14).

At $\sigma = 1$ retain these powers. Completing the prime-power sums for $\mathrm{N}\mathfrak{p} \le y$ changes the logarithm by $O_K(y^{-1/3})$. Indeed primes with norm exceeding $\sqrt{y}$ contribute $O_K(\sum_{\mathrm{N}\mathfrak{p}>\sqrt{y}}(\mathrm{N}\mathfrak{p})^{-2}) = O_K(y^{-1/2})$. For primes of norm at most $\sqrt{y}$, each geometric tail is at most $2/y$, and there are $O_K(\sqrt{y})$ such primes. Enlarging B once more and exponentiating proves (4.15). $\square$

5. Gál sums over squarefree ideals

The resonator will contain many ideals, but its prime divisors must remain small. We also need control of the products of four resonator ideals. We obtain both properties by choosing only a few primes from each of several disjoint blocks.

For integral ideals $\mathfrak{m},\mathfrak{n}$, write $(\mathfrak{m},\mathfrak{n})$ for their ideal gcd and $[\mathfrak{m},\mathfrak{n}]$ for their ideal lcm. For a finite set $\mathcal{A}$ of squarefree ideals and $s > 0$, put

$$S_s(\mathcal{A}) := \sum_{\mathfrak{m},\mathfrak{n}\in\mathcal{A}}\left(\frac{\mathrm{N}(\mathfrak{m},\mathfrak{n})}{\mathrm{N}[\mathfrak{m},\mathfrak{n}]}\right)^s.$$

The quotient in this kernel is the reciprocal of the norm of the symmetric difference of the two prime supports. Define also

$$E_\square(\mathcal{A}) := \#\{(\mathfrak{a}_1,\mathfrak{a}_2,\mathfrak{a}_3,\mathfrak{a}_4)\in\mathcal{A}^4 : \mathfrak{a}_1\mathfrak{a}_2\mathfrak{a}_3\mathfrak{a}_4 \text{ is a square}\}. \tag{5.1}$$

This counts the terms that survive in the fourth moment of the resonator. If squarefree products are identified with their support vectors, $E_\square$ is additive energy in a vector space over $\mathbb{F}_2$. We use the same definition for families of subsets of a finite set.

Lemma 5.1. *Let $P \ge 1$ and $0 \le J \le P/2$ be integers. Let $\mathcal{A}(P,J)$ be the family of subsets of a P-element set having at most J elements.*

(i) *We have*

$$E_\square(\mathcal{A}(P,J)) \le |\mathcal{A}(P,J)|^2(J+1)^2\exp\left(4J+\frac{4J^2}{P}\right). \tag{5.2}$$

(ii) *If the underlying sets of the factors are disjoint, then*

$$E_\square\left(\prod_{k=1}^t \mathcal{A}(P_k,J_k)\right) = \prod_{k=1}^t E_\square(\mathcal{A}(P_k,J_k)). \tag{5.3}$$

Proof. For the count within one layer, let S_j be the family of j-element subsets. If D has $2r$ elements, the number of ordered pairs $(A,B)\in S_j^2$ with $A\triangle B = D$ is

$$\binom{2r}{r}\binom{P-2r}{j-r}.$$

Indeed, r elements of D belong to $A\setminus B$. The rest belong to $B\setminus A$. Their common intersection has $j-r$ elements outside D. For odd $|D|$, or $|D| > 2j$, the count is zero. Pairing equal symmetric differences therefore gives

$$E_\square(S_j) = \sum_{r=0}^j \binom{P}{2r}\binom{2r}{r}^2\binom{P-2r}{j-r}^2. \tag{5.4}$$

For $1 \le j \le P/2$, use

$$\binom{P}{2r} \le \frac{P^{2r}}{(2r)!}, \qquad \binom{2r}{r} \le 4^r, \qquad \binom{P-2r}{j-r} \le \frac{P^{j-r}}{(j-r)!}, \qquad \binom{P}{j} \ge \frac{(P-j)^j}{j!}.$$

Since $-\log(1-j/P) \le 2j/P$, these inequalities imply

$$\begin{aligned}\frac{E_\square(S_j)}{|S_j|^2} &\le \left(\frac{P}{P-j}\right)^{2j} \sum_{r=0}^{j} \frac{16^r (j!/(j-r)!)^2}{(2r)!} \\ &\le e^{4j^2/P} \sum_{r=0}^{j} \frac{(4j)^{2r}}{(2r)!} \le e^{4j+4j^2/P}.\end{aligned}$$

The same bound holds for $j = 0$.

The ball also contains quadruples from different layers. To retain these terms, work in $G = \mathbb{F}_2^P$ and define $\widehat{f}(\xi) = \sum_{x \in G} f(x)(-1)^{x\cdot\xi}$. For subsets $B_1, \dots, B_4$ of G, Fourier inversion and Hölder's inequality give

$$\begin{aligned}\#\{(x_i) \in \textstyle\prod_i B_i : \sum_i x_i = 0\} &= |G|^{-1} \sum_{\xi \in G} \prod_{i=1}^{4} \widehat{\mathbf{1}_{B_i}}(\xi) \\ &\le \prod_{i=1}^{4} E_\square(B_i)^{1/4}.\end{aligned}$$

Here $|G|^{-1} \sum_\xi |\widehat{\mathbf{1}_B}(\xi)|^4 = E_\square(B)$. Expand the ball into its disjoint layers and apply this inequality. Then

$$\begin{aligned}E_\square(\mathcal{A}(P,J)) &\le \left(\sum_{j=0}^{J} E_\square(S_j)^{1/4}\right)^4 \\ &\le e^{4J+4J^2/P} \left(\sum_{j=0}^{J} |S_j|^{1/2}\right)^4 \\ &\le e^{4J+4J^2/P} (J+1)^2 \left(\sum_{j=0}^{J} |S_j|\right)^2.\end{aligned}$$

This proves the first assertion. In a product of disjoint blocks, the square condition holds in each block separately. Its count therefore factorizes, proving the second assertion. $\square$

We next estimate the gain from one block. Two products in the block may share most of their primes. The kernel then charges only the primes that differ. The following calculation balances the number of such pairs against their weights.

Lemma 5.2. *Let $\mathscr{P}$ have P elements, with weights $0 < w_p < 1$. Put*

$$H := \sum_{p \in \mathscr{P}} w_p, \qquad W := \max_{p \in \mathscr{P}} w_p, \qquad S_{\mathscr{P}}(J) := \sum_{A,B \in \mathcal{A}(P,J)} \prod_{p \in A \triangle B} w_p.$$

Suppose

$$1 \le r \le J/2, \qquad J \le P/3, \qquad (J+2r)W/H \le 1/2.$$

Then, with an absolute implied constant,

$$\begin{aligned}\log \frac{S_{\mathscr{P}}(J)}{|\mathcal{A}(P,J)|} &\ge r \log\left(\frac{e^2 J H^2}{P r^2}\right) \\ &\quad - O\left(r\frac{(J+2r)W}{H} + \frac{r^2}{J} + \frac{rJ}{P} + \log(r+1)\right). \qquad (5.5)\end{aligned}$$

Consequently, if

$$r \longrightarrow \infty, \qquad r/J \longrightarrow 0, \qquad J/P \longrightarrow 0, \qquad (J+r^2)W/H \longrightarrow 0,$$

then, uniformly under these hypotheses,

$$\log \frac{S_{\mathscr{P}}(J)}{|\mathcal{A}(P,J)|} \geq r \log\left(\frac{e^2 J H^2}{P r^2}\right) + o(r). \tag{5.6}$$

Proof. Restrict the sum to pairs $A = D\dot{\cup}X$ and $B = D\dot{\cup}Y$, where $|D| = J-r$, $|X| = |Y| = r$, and D, X, Y are pairwise disjoint. We need a lower bound for the weighted choices of X and Y after some primes have been excluded.

Let E be an excluded set with $|E| \leq J+r$. After j distinct entries have been chosen, the total weight available for the next entry is at least $H - (|E|+j)W$. Thus the total weight of ordered distinct r-tuples outside E lies between

$$\prod_{j=0}^{r-1}\bigl(H - (|E|+j)W\bigr) \quad \text{and} \quad H^r.$$

By the hypothesis $(J+2r)W/H \leq 1/2$, every factor in this product is positive. Using $\log(1-x) \geq -2x$ for $0 \leq x \leq 1/2$ gives the quantitative lower bound

$$\prod_{j=0}^{r-1}\bigl(H - (|E|+j)W\bigr) \geq H^r \exp\left(-O\left(\frac{r(J+2r)W}{H}\right)\right).$$

Dividing by $r!$ gives the corresponding sum over unordered subsets. Apply this first with $E = D$, then with $E = D \cup X$. Since D has $\binom{P}{J-r}$ possible values,

$$S_{\mathscr{P}}(J) \geq \binom{P}{J-r}\frac{H^{2r}}{(r!)^2}\exp\left(-O\left(\frac{r(J+2r)W}{H}\right)\right).$$

Since $J \leq P/3$ and $r \leq J/2$, comparison of successive binomial coefficients and a logarithmic expansion give

$$|\mathcal{A}(P,J)| \leq 2\binom{P}{J}, \qquad \frac{\binom{P}{J-r}}{\binom{P}{J}} \geq \left(\frac{J}{P}\right)^r \exp\left(-O\left(\frac{r^2}{J} + \frac{rJ}{P}\right)\right).$$

For the second bound, write the ratio as $\prod_{j=0}^{r-1}(J-j)/(P-J+j+1)$ and take logarithms. Stirling's formula now gives

$$\begin{aligned}\log \frac{S_{\mathscr{P}}(J)}{|\mathcal{A}(P,J)|} \geq\ & r\log\frac{J}{P} + 2r\log H - 2r\log r + 2r \\ & - O\left(\frac{r(J+2r)W}{H} + \frac{r^2}{J} + \frac{rJ}{P} + \log(r+1)\right).\end{aligned}$$

This proves (5.5). Under the asymptotic hypotheses of the lemma, every term in the error is $o(r)$, uniformly, and (5.6) follows. ☐

Proposition 5.3. *Fix S. For every sufficiently large integer N, there is a set $\mathcal{M}$ of N squarefree integral ideals coprime to S with the following properties.*

(i) *Its Gál sum satisfies*

$$S_{1/2}(\mathcal{M}) \geq N\exp\left\{(2+o(1))\sqrt{\frac{\log N \log_3 N}{\log_2 N}}\right\}. \tag{5.7}$$

More generally, uniformly for A in any fixed compact subset of $[0,\infty)$,

$$S_{1/2+A/\log_2 N}(\mathcal{M}) \geq N\exp\left\{(2e^{-A}+o(1))\sqrt{\frac{\log N \log_3 N}{\log_2 N}}\right\}. \tag{5.8}$$

(ii) *Its prime support and square energy satisfy*

$$y_{\mathcal{M}} := \max_{\substack{\mathfrak{m}\in\mathcal{M}\\ \mathfrak{p}\mid\mathfrak{m}}} \mathrm{N}\mathfrak{p} \le (\log N)^{1+o(1)}, \tag{5.9}$$

and

$$E_{\square}(\mathcal{M}) \le N^{2+o(1)}. \tag{5.10}$$

Proof. The construction uses blocks of geometrically increasing prime norms. The number of primes selected from the kth block decreases like k^{-2}. This makes the logarithm of the cardinality and the gain in the kernel depend on the same harmonic sum.

First fix $1 < u \le e$, $0 < \gamma < 1$, and $0 < \eta < 1/2$. Write $T_i = \log_i N$ for $1 \le i \le 4$, and put $K_N = \lfloor T_2^{\gamma} \rfloor$. Let $\mathscr{P}_k$ consist of the prime ideals outside S whose norms belong to

$$(u^k T_1 T_2, u^{k+1} T_1 T_2], \qquad 1 \le k \le K_N.$$

Set $P_k = |\mathscr{P}_k|$ and

$$H_k := \sum_{\mathfrak{p}\in\mathscr{P}_k} (\mathrm{N}\mathfrak{p})^{-1/2}, \qquad W_k := \max_{\mathfrak{p}\in\mathscr{P}_k} (\mathrm{N}\mathfrak{p})^{-1/2}.$$

For fixed u and γ, the block endpoints satisfy $\log(u^k T_1 T_2) = T_2 + o(T_2)$ uniformly for $k \le K_N$. Thus the prime ideal theorem (and partial summation) may be applied uniformly over all the blocks, giving

$$P_k = u^k (u-1) T_1 \{1 + o(1)\}, \tag{5.11}$$

$$H_k = 2u^{k/2} (\sqrt{u} - 1) \sqrt{T_1/T_2} \{1 + o(1)\}. \tag{5.12}$$

Here the logarithm of every norm in a block is $T_2 + o(T_2)$. In particular, with $\lambda(u) = 4(\sqrt{u}-1)/(\sqrt{u}+1)$, we have

$$\frac{H_k^2}{P_k} = \frac{\lambda(u) + o(1)}{T_2}, \qquad \frac{W_k}{H_k} \ll_u \frac{u^{-k}}{T_1}. \tag{5.13}$$

For a fixed $a > 0$, put $J_k(a) = \lfloor a T_1/(k^2 T_3) \rfloor$. Let $\mathcal{M}(a)$ contain all products obtained by choosing at most $J_k(a)$ primes from each $\mathscr{P}_k$. We claim that

$$\log |\mathcal{M}(a)| = (a\gamma \log u + o(1)) \log N. \tag{5.14}$$

To verify the cardinality, first note the uniform estimate

$$\log \sum_{\nu=0}^{J} \binom{P}{\nu} = J \log \frac{P}{J} + J + O\left(\log(J+1) + \frac{J^2}{P}\right) \qquad (1 \le J,\ J/P = o(1)). \tag{5.15}$$

Indeed, $\binom{P}{J-\ell}/\binom{P}{J} \le (J/(P-J+1))^{\ell}$, so the sum is $\binom{P}{J}(1 + O(J/P))$. Apply Stirling's formula to

$$\log \binom{P}{J} = J \log P - \log J! + \sum_{\nu=0}^{J-1} \log(1 - \nu/P).$$

The last sum is $O(J^2/P)$, proving the estimate.

For our quotas,

$$\log \frac{P_k}{J_k(a)} = k \log u + 2 \log k + T_4 + O_{u,a}(1).$$

The first term provides the main contribution, since

$$\sum_{k \le K_N} k J_k(a) = \frac{a T_1}{T_3} \sum_{k \le K_N} \frac{1}{k} + O(K_N^2) = (a\gamma + o(1)) T_1.$$

The remaining terms satisfy

$$\sum_{k\le K_N} J_k(a)(1+\log k) = O_{u,a}(T_1/T_3)+o(T_1),$$

$$T_4\sum_{k\le K_N} J_k(a) = O_{u,a}(T_1T_4/T_3)+o(T_1)=o(T_1),$$

$$\sum_{k\le K_N}\log(J_k(a)+1)\le K_NT_2=o(T_1),$$

$$\sum_{k\le K_N}\frac{J_k(a)^2}{P_k}\ll_{u,a}\frac{T_1}{T_3^2}\sum_{k\ge1}\frac{u^{-k}}{k^4}=o(T_1).$$

Also $K_N^2=o(T_1)$, so the floor errors are negligible. Summing (5.15) proves (5.14).

To bring the cardinality close to N, we adjust the quotas. Put $a_\pm=(1\pm\eta)/(\gamma\log u)$. For large N, $|\mathcal{M}(a_-)|<N<|\mathcal{M}(a_+)|$. Starting with the lower quotas, increase one quota by one at each step, without exceeding the upper quotas. Stop before the cardinality first exceeds N. Denote the resulting family by $\mathcal{M}_*$ and its cardinality by M. Its quotas satisfy $J_k(a_-)\le J_k\le J_k(a_+)$, and

$$M\le N,\qquad N/M\le 1+\max_{k\le K_N}P_k\le(\log N)^{1+o(1)}. \tag{5.16}$$

For the middle inequality, a single quota increase changes the cardinality by at most

$$\frac{\sum_{j\le J+1}\binom{P}{j}}{\sum_{j\le J}\binom{P}{j}}\le 1+\frac{\binom{P}{J+1}}{\binom{P}{J}}\le 1+P.$$

We estimate the central and shifted kernels together. Fix $A_0\ge0$ and let $0\le A\le A_0$. Define

$$H_k(A):=\sum_{\mathfrak{p}\in\mathscr{P}_k}(\mathrm{N}\mathfrak{p})^{-1/2-A/T_2},\qquad W_k(A):=\max_{\mathfrak{p}\in\mathscr{P}_k}(\mathrm{N}\mathfrak{p})^{-1/2-A/T_2}.$$

Since $\log\mathrm{N}\mathfrak{p}=T_2+o(T_2)$ uniformly on the blocks,

$$H_k(A)=e^{-A}H_k\{1+o(1)\},\qquad W_k(A)/H_k(A)=\{1+o(1)\}W_k/H_k. \tag{5.17}$$

Thus a shift of A/T_2 multiplies the available weight in each block by $e^{-A}+o(1)$. Choose

$$\alpha=\sqrt{a_-\lambda(u)},\qquad r_k(A)=\left\lfloor\frac{e^{-A}\alpha}{k}\sqrt{\frac{T_1}{T_2T_3}}\right\rfloor.$$

These values balance the gain from the weights with the cost of choosing different primes. We check the hypotheses of the preceding lemma, uniformly for $k\le K_N$ and $0\le A\le A_0$:

$$\min_k r_k(A)\gg_{u,\gamma,\eta,A_0}\frac{\sqrt{T_1}}{T_2^{\gamma+1/2}\sqrt{T_3}}\longrightarrow\infty,$$

$$\max_k\frac{r_k(A)}{J_k}\ll_{u,\gamma,\eta,A_0}\frac{K_N\sqrt{T_3}}{\sqrt{T_1T_2}}\longrightarrow0,$$

$$\max_k\frac{J_k}{P_k}\ll_{u,\gamma,\eta}T_3^{-1}\longrightarrow0,$$

$$\frac{(J_k+r_k(A)^2)W_k(A)}{H_k(A)}\ll_{u,\gamma,\eta,A_0}\frac{u^{-k}}{k^2T_3}+\frac{u^{-k}}{k^2T_2T_3}=o(1).$$

All four $o(1)$ statements are uniform in $k\le K_N$ and $0\le A\le A_0$. In particular, the quantitative error in (5.5), divided by $r_k(A)$, tends to zero uniformly. Since $\min_k r_k(A)\to\infty$, summing the block errors therefore contributes

$$o\left(\sum_{k\le K_N} r_k(A)\right)$$

uniformly for $0 \le A \le A_0$. Moreover, (5.13) and the lower quota bound imply

$$\frac{e^2 J_k H_k(A)^2}{P_k r_k(A)^2} \ge e^2\{1+o(1)\}.$$

The kernel factorizes across the disjoint blocks. Apply Lemma 5.2 and sum its logarithmic bounds to obtain

$$\begin{aligned}\log\frac{S_{1/2+A/T_2}(\mathcal{M}_*)}{M} &\ge (2+o(1))\sum_{k\le K_N} r_k(A)\\ &= \left(2e^{-A}\sqrt{\gamma(1-\eta)\frac{\lambda(u)}{\log u}}+o(1)\right)\sqrt{\frac{T_1T_3}{T_2}}. \end{aligned}\tag{5.18}$$

Here $\sum_{k\le K_N} 1/k = \gamma T_3 + O(1)$, and the total floor error $O(K_N)$ is $o(\sqrt{T_1T_3/T_2})$.

To obtain exactly N ideals, first double the cardinality as often as possible using new prime divisors. Put $h = \lfloor \log(N/M)/\log 2\rfloor$. By (5.16), $h = O(T_2)$. Choose distinct prime ideals $\mathfrak{q}_1,\dots,\mathfrak{q}_h$ outside S and the original blocks, with norms at most $T_2^{O_K(1)}$. The prime ideal theorem supplies these primes below the first block. Let $\mathcal{C}_h$ be the set of divisors of their product, and put $\mathcal{M}_1 = \mathcal{C}_h\mathcal{M}_*$. Then $N/2 < |\mathcal{M}_1| \le N$. For every $s>0$,

$$\frac{S_s(\mathcal{C}_h)}{|\mathcal{C}_h|} = \prod_{\ell=1}^{h}(1+(\mathrm{N}\mathfrak{q}_\ell)^{-s}) \ge 1,$$

and disjointness of the supports gives

$$\frac{S_s(\mathcal{M}_1)}{|\mathcal{M}_1|} = \frac{S_s(\mathcal{C}_h)}{|\mathcal{C}_h|}\frac{S_s(\mathcal{M}_*)}{M} \ge \frac{S_s(\mathcal{M}_*)}{M}. \tag{5.19}$$

Now add unused elements of $\mathcal{M}(a_+)$ until the cardinality is N. There are enough: the auxiliary cube uses primes disjoint from all original blocks, so $\mathcal{M}(a_+)\cap\mathcal{M}_1 = \mathcal{M}_*$. Hence the number of available new elements is $|\mathcal{M}(a_+)| - M > N - M$, whereas at most $N - |\mathcal{M}_1| \le N - M$ elements are needed. Call the completed set $\mathcal{M}$. Positivity of the kernel yields

$$\frac{S_s(\mathcal{M})}{N} \ge \frac{|\mathcal{M}_1|}{N}\frac{S_s(\mathcal{M}_1)}{|\mathcal{M}_1|} > \frac{1}{2}\frac{S_s(\mathcal{M}_*)}{M}.$$

Thus completing the cardinality costs only a bounded factor.

We must also check that the completion has small square energy. Lemma 5.1 gives

$$\begin{aligned}\log\frac{E_\square(\mathcal{M}(a_+))}{|\mathcal{M}(a_+)|^2} &\ll \sum_{k\le K_N}\left(J_k(a_+)+\frac{J_k(a_+)^2}{P_k}+\log(J_k(a_+)+1)\right)\\ &\ll_{u,\gamma,\eta} T_1/T_3 + T_1/T_3^2 + K_N T_2 = o(T_1). \end{aligned}\tag{5.20}$$

All quotas are at most $P_k/2$ for large N, as the lemma requires. Every element of $\mathcal{M}$ belongs to $\mathcal{U} = \mathcal{C}_h\mathcal{M}(a_+)$. The full cube $\mathcal{C}_h$ has square energy 2^{3h}: three entries determine the fourth. Therefore

$$E_\square(\mathcal{M}) \le E_\square(\mathcal{U}) = 2^{3h}E_\square(\mathcal{M}(a_+)) \le N^{2+2\eta+o(1)}.$$

We used $2^h \le N/M \le (\log N)^{1+o(1)}$ and $|\mathcal{M}(a_+)| = N^{1+\eta+o(1)}$.

Finally, let the fixed parameters approach their limiting values by diagonal selection. For each integer $j\ge 3$, take

$$u_j = e^{1/j}, \qquad \gamma_j = 1-1/j, \qquad \eta_j = 1/j.$$

All estimates above hold for these fixed parameters, uniformly for $0 \le A \le j$. More explicitly, for each fixed j every $o(1)$ occurring in the block estimates may be chosen so that its supremum over $0 \le A \le j$ and $1 \le k \le K_N$ tends to zero as $N \to \infty$. Thus one may choose a single threshold N_j which works simultaneously for all these parameters. Choose increasing thresholds N_j so that, for $N \ge N_j$, every normalized error is at most $1/j$, and $E_\square(\mathcal{U}) \le N^{2+5/j}$. We may also require

$$\frac{\log 9}{\log N}\left(h + \sum_{k \le K_N} J_k(a_+)\right) \le \frac{1}{j};$$

for each fixed j, the sum in parentheses is $O_j(\log N/\log_3 N + \log_2 N) = o(\log N)$. This additional condition records the degree bound used in Remark 5.4. On $N_j \le N < N_{j+1}$ use the jth construction. Since $\lambda(u)/\log u \to 1$ as $u \downarrow 1$, the coefficient in (5.18) tends to $2e^{-A}$, uniformly on every fixed compact A-interval. This proves (5.8) and its case $A = 0$, namely (5.7). The energy bound follows from $j = j(N) \to \infty$.

The largest norm in an original block is at most $u_j^{K_N+1} T_1 T_2$. Its logarithm satisfies

$$\frac{\log(u_j^{K_N+1} T_1 T_2)}{T_2} = 1 + \frac{T_3}{T_2} + \frac{K_N + 1}{jT_2} \le 1 + o(1) + \frac{1}{j}.$$

The auxiliary primes have smaller norms, proving (5.9). All ideals are squarefree and coprime to S by construction. $\square$

Remark 5.4 (Energy and hypercontractivity). For a finite nonempty set $\mathcal{A}$ of squarefree ideals, put

$$\Delta(\mathcal{A}) = \max_{\mathfrak{m} \in \mathcal{A}} \#\{\mathfrak{p} : \mathfrak{p} \mid \mathfrak{m}\}.$$

Let $x_\mathfrak{p}$ be independent uniform signs indexed by the union of its prime supports, and define

$$F_\mathcal{A}(x) = \sum_{\mathfrak{m} \in \mathcal{A}} \prod_{\mathfrak{p} \mid \mathfrak{m}} x_\mathfrak{p}.$$

This multilinear polynomial has degree at most $\Delta(\mathcal{A})$. Orthogonality of the sign monomials gives

$$\mathbb{E}F_\mathcal{A}^2 = |\mathcal{A}|, \qquad \mathbb{E}F_\mathcal{A}^4 = E_\square(\mathcal{A}).$$

Bonami hypercontractivity [31, Theorem 9.21, with $q = 4$] states that $\|F_\mathcal{A}\|_4 \le 3^{\Delta(\mathcal{A})/2}\|F_\mathcal{A}\|_2$. Consequently,

$$E_\square(\mathcal{A}) \le 9^{\Delta(\mathcal{A})}|\mathcal{A}|^2. \tag{5.21}$$

For the completed set in Proposition 5.3,

$$\Delta(\mathcal{M}) \le h + \sum_{k \le K_N} J_k(a_+) \le h + \frac{\pi^2 a_+}{6}\frac{\log N}{\log_3 N}.$$

The thresholds chosen above ensure $\Delta(\mathcal{M}) = o(\log N)$, so (5.21) also proves $E_\square(\mathcal{M}) \le N^{2+o(1)}$ directly on the completed set. Thus the energy estimate is a consequence of sparsity. The separate block and truncation estimates establish that this sparsity is compatible with the required Gál gain, prime support, and approximate functional equation. Lemma 5.1 provides a self-contained combinatorial route to the energy bound used in the proof.

The approximate functional equation only sees pairs whose symmetric difference has bounded norm. The next lemma shows that these pairs already carry the required Gál mass.

Lemma 5.5. *Fix $\varepsilon > 0$, and let $\mathcal{M}$ be supplied by Proposition 5.3.*

(i) *If $N \le X^{1/4}$, then*

$$\sum_{\substack{\mathfrak{m},\mathfrak{n} \in \mathcal{M} \\ \mathrm{N}[\mathfrak{m},\mathfrak{n}]/\mathrm{N}(\mathfrak{m},\mathfrak{n}) \le X^\varepsilon}} \sqrt{\frac{\mathrm{N}(\mathfrak{m},\mathfrak{n})}{\mathrm{N}[\mathfrak{m},\mathfrak{n}]}} \ge N \exp\left\{(2 + o(1))\sqrt{\frac{\log N \log_3 N}{\log_2 N}}\right\}.$$

(ii) *Suppose $N = X^{\beta+o(1)}$, where $0 < \beta \le 1/4$ is fixed. Uniformly for A in any fixed compact subset of* $[0,\infty)$,

$$\sum_{\substack{\mathfrak{m},\mathfrak{n}\in\mathcal{M}\\ \mathrm{N}[\mathfrak{m},\mathfrak{n}]/\mathrm{N}(\mathfrak{m},\mathfrak{n})\le X^{\varepsilon}}} \left(\frac{\mathrm{N}(\mathfrak{m},\mathfrak{n})}{\mathrm{N}[\mathfrak{m},\mathfrak{n}]}\right)^{1/2+A/\log_2 X}$$
$$\ge N\exp\left\{(2e^{-A}+o(1))\sqrt{\frac{\log N\log_3 N}{\log_2 N}}\right\}.$$

Proof. For an omitted pair put $d = \mathrm{N}[\mathfrak{m},\mathfrak{n}]/\mathrm{N}(\mathfrak{m},\mathfrak{n}) > X^{\varepsilon}$. Then $d^{-1/2} \le X^{-\varepsilon/6}d^{-1/3}$. Fix $\mathfrak{m}$ and enlarge the sum over $\mathfrak{n}$ to all squarefree products of primes with norms at most $y_{\mathcal{M}}$. At each prime, the two choices contribute $1 + (\mathrm{N}\mathfrak{p})^{-1/3}$, regardless of whether $\mathfrak{p}$ divides $\mathfrak{m}$. Hence

$$\sum_{\mathfrak{n}\in\mathcal{M}} d^{-1/3} \le \prod_{\mathrm{N}\mathfrak{p}\le y_{\mathcal{M}}} (1+(\mathrm{N}\mathfrak{p})^{-1/3}) \le \exp\{O_K(y_{\mathcal{M}}^{2/3})\}.$$

By (5.9), the total omitted central mass is at most

$$NX^{-\varepsilon/6}\exp\{(\log N)^{2/3+o(1)}\} = o(N)$$

in either range for N. Indeed $N \le X^{1/4+o(1)}$ in the applications here, so $(\log N)^{2/3+o(1)} = o(\log X)$ and the exponential factor cannot offset the fixed power $X^{-\varepsilon/6}$. The omitted shifted mass is no larger. Subtracting this bound from (5.7) proves the first assertion.

For the second, put $B_X = A\log_2 N/\log_2 X$. Then $B_X = A + o(1)$ uniformly for bounded A, and $B_X/\log_2 N = A/\log_2 X$ exactly. Apply the compact-uniform estimate (5.8) with B_X. Since $e^{-B_X} = e^{-A} + o(1)$, its main term is the one asserted above. Subtract the same $o(N)$ tail to finish the proof. □

Fix one set $\mathcal{M}$ from the proposition for every sufficiently large N. The two bounds in its second assertion can be recorded with one function. Define

$$\omega(t) := \sup_{N\ge t}\max\left\{0, \frac{\log y_{\mathcal{M}}}{\log_2 N} - 1, \frac{\log E_{\Box}(\mathcal{M})}{\log N} - 2\right\}. \tag{5.22}$$

Then $\omega(t) \to 0$, and the chosen sets satisfy $y_{\mathcal{M}} \le (\log N)^{1+\omega(N)}$ and $E_{\Box}(\mathcal{M}) \le N^{2+\omega(N)}$. These are the uniform bounds used in the fourth-moment argument.

6. Large values near the central point

We prove all three assertions of Theorem 1.1 together. The same argument covers the central point by taking $A = 0$. The resonator assigns extra weight to characters whose values agree on many small prime ideals. The Gál sum measures the resulting gain in the first moment. A fourth-moment estimate then shows that this gain cannot come from too few characters.

Fix a nonzero function $W \in C_c^{\infty}((1,2))$ with $0 \le W \le 1$, and put

$$I_W := \int_0^{\infty} W(t)\,dt > 0.$$

For a finite set $\mathcal{M}$ of ideals, write

$$R_{\mathfrak{a}} := \sum_{\mathfrak{m}\in\mathcal{M}} \chi_{\mathfrak{a}}(\mathfrak{m}).$$

These sums are real because the characters are quadratic.

Lemma 6.1. *Assume* GRH_K*. Fix $0 < \beta < 1/4$, and put $N = \lfloor X^{\beta}\rfloor$. Let $\mathcal{M}$ be an N-element set of squarefree ideals coprime to S. Suppose that a fixed function $\omega(t) \to 0$ satisfies*

$$y_{\mathcal{M}} \le (\log N)^{1+\omega(N)}, \qquad E_{\Box}(\mathcal{M}) \le N^{2+\omega(N)}.$$

Then

$$\sum_{\mathfrak{a}\in\Omega} W(\mathrm{N}\mathfrak{a}/X)|R_{\mathfrak{a}}|^4 \ll_{K,\Omega,W,\beta} XN^{2+o(1)}. \tag{6.1}$$

Proof. The fourth power counts quadruples of resonator ideals. The character average distinguishes those whose product is a square. Since $|R_{\mathfrak{a}}|^4 = R_{\mathfrak{a}}^4$, Proposition 3.1 gives

$$\begin{aligned}\sum_{\mathfrak{a}\in\Omega} & W(\mathrm{N}\mathfrak{a}/X)|R_{\mathfrak{a}}|^4 \\ &= \kappa_\Omega I_W X \sum_{\substack{\mathfrak{m}_1,\dots,\mathfrak{m}_4\in\mathcal{M}\\ \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3\mathfrak{m}_4=\square}} h_K(\mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3\mathfrak{m}_4) + O\left(X^{1/2+\varepsilon+o(1)}N^4\right).\end{aligned} \tag{6.2}$$

Here Lemma 3.2 applies with $\mathfrak{u} = (1)$. Indeed, for every fixed $\varepsilon > 0$, the support hypothesis gives $y_{\mathcal{M}} \le (\log X)^{1+\varepsilon/4}$ once X is large. The integral main term is at most $\kappa_\Omega I_W X E_\square(\mathcal{M})$, since $0 < h_K \le 1$. Choose $0 < \varepsilon < 1/2 - 2\beta$. The error divided by XN^2 is

$$O\left(X^{-1/2+\varepsilon+o(1)}N^2\right) = O\left(X^{-1/2+\varepsilon+2\beta+o(1)}\right) = o(1).$$

The assumed energy bound now proves the lemma. $\square$

Proof of Theorem 1.1. Fix $A_0 \ge 0$ and $0 < \beta < 1/4$. Throughout the proof, $0 \le A \le A_0$ and

$$N = \lfloor X^\beta \rfloor, \qquad \delta = \frac{A}{\log_2 X}, \qquad s = \frac{1}{2} + \delta.$$

Take $\mathcal{M}$ from Proposition 5.3, with the common support and energy bounds in (5.22). Define

$$S_0 := \sum_{\mathfrak{a}\in\Omega} W(\mathrm{N}\mathfrak{a}/X)R_{\mathfrak{a}}^2, \qquad S_1(A) := \sum_{\mathfrak{a}\in\Omega} W(\mathrm{N}\mathfrak{a}/X)L(s,\chi_{\mathfrak{a}})R_{\mathfrak{a}}^2.$$

Write $S_1 = S_1(0)$ at the central point.

For squarefree ideals $\mathfrak{m},\mathfrak{n}$, their product is a square if and only if $\mathfrak{m} = \mathfrak{n}$. Expand S_0 and apply Proposition 3.1 and Lemma 3.2. This gives

$$S_0 = \kappa_\Omega I_W X \sum_{\mathfrak{m}\in\mathcal{M}} h_K(\mathfrak{m}^2) + O\left(X^{1/2+\varepsilon+o(1)}N^2\right).$$

We fix $0 < \varepsilon < \min\{1/4-\beta, 1/4\}$ for the rest of the moment calculation. Since $h_K \le 1$, we obtain

$$S_0 \le (\kappa_\Omega I_W + o(1))XN. \tag{6.3}$$

The first moment contains a second ideal sum from the approximate functional equation. Its weights must retain their decay when we apply the character-average error bound. For $t > 0$, put

$$\mathcal{Q}(t) := |D_K|\mathrm{N}\mathfrak{c}_\Omega\, t.$$

The two test functions in (4.2) are

$$w^+_{\mathfrak{k},X,A}(t) := W(t/X)V_{K,\delta}\left(\frac{\mathrm{N}\mathfrak{k}}{\sqrt{\mathcal{Q}(t)}}\right),$$

$$w^-_{\mathfrak{k},X,A}(t) := W(t/X)\mathcal{Q}(t)^{-\delta}\frac{\gamma_K(1/2-\delta)}{\gamma_K(1/2+\delta)}V_{K,-\delta}\left(\frac{\mathrm{N}\mathfrak{k}}{\sqrt{\mathcal{Q}(t)}}\right).$$

For large X, we have $0 \le \delta \le \delta_0 < 1/4$, where δ_0 is fixed. Differentiating $\mathrm{N}\mathfrak{k}/\sqrt{\mathcal{Q}(t)}$ multiplies its logarithmic derivatives by fixed powers of t^{-1}. Moreover, $t^j(d/dt)^j\mathcal{Q}(t)^{-\delta} = O_j(X^{-\delta})$ on $[X,2X]$, uniformly in this range of δ. Thus (4.5) gives, for every fixed $B > 0$,

$$\|w^+_{\mathfrak{k},X,A}\|_{*,X} \ll_{K,W,A_0,B} (1+\mathrm{N}\mathfrak{k}/\sqrt{X})^{-B},$$

$$\|w^-_{\mathfrak{k},X,A}\|_{*,X} \ll_{K,W,A_0,B} X^{-\delta}(1+\mathrm{N}\mathfrak{k}/\sqrt{X})^{-B}.$$

The ideal-counting bound $\#\{\mathfrak{k} : \mathrm{N}\mathfrak{k} \le t\} \ll_K t$ and partial summation imply

$$\sum_{\mathfrak{k}} (\mathrm{N}\mathfrak{k})^{-a}(1+\mathrm{N}\mathfrak{k}/\sqrt{X})^{-B} \ll_{K,B} X^{(1-a)/2} \qquad (1/4 \le a \le 3/4, B \ge 2).$$

Consequently,

$$\sum_{\mathfrak{k}} (\mathrm{N}\mathfrak{k})^{-1/2-\delta}\|w^+_{\mathfrak{k},X,A}\|_{*,X} \ll X^{1/4-\delta/2} \le X^{1/4}, \tag{6.4}$$

and

$$\sum_{\mathfrak{k}} (\mathrm{N}\mathfrak{k})^{-1/2+\delta}\|w^-_{\mathfrak{k},X,A}\|_{*,X} \ll X^{-\delta}X^{1/4+\delta/2} \le X^{1/4}. \tag{6.5}$$

These estimates concern the seminorms themselves, so they apply directly to the errors in Proposition 3.1.

We may first truncate both ideal sums at $\mathrm{N}\mathfrak{k} \le X^2$. Indeed, the decay estimate with B sufficiently large gives

$$\sum_{\mathrm{N}\mathfrak{k} > X^2} (\mathrm{N}\mathfrak{k})^{-1/2+\delta}(1+\mathrm{N}\mathfrak{k}/\sqrt{X})^{-B} \ll X^{B/2+2(1/2+\delta-B)}.$$

Since $\#\Omega(X) \ll X$ and $|R_{\mathfrak{a}}| \le N$, the contribution of these tails to $S_1(A)$ is $O(X^{-100})$ after increasing B. For the remaining ideals, Lemma 3.2 applies with $\mathfrak{u} = \mathfrak{k}$ and $\mathfrak{v} = \mathfrak{m}\mathfrak{n}$, taking $B_0 = 2$. More explicitly, if $\mathfrak{k}\mathfrak{m}\mathfrak{n} = \mathfrak{l}_0\mathfrak{l}_1^2$ with $\mathfrak{l}_0$ squarefree, there is a constant $C_{K,\varepsilon}$ such that

$$\sup_{\substack{\mathrm{N}\mathfrak{k} \le X^2 \\ \mathfrak{m},\mathfrak{n} \in \mathcal{M}}} F_\varepsilon(\mathfrak{l}_0)G_\varepsilon(\mathfrak{l}_1) \le \exp\{C_{K,\varepsilon}(\log X)^{1-\varepsilon/2}\} = X^{o(1)}.$$

This bound uses only the common prime support of $\mathcal{M}$, so it also holds uniformly when $\mathcal{M}$ varies with $0 \le A \le A_0$. It can therefore be taken outside the whole truncated ideal sum and both resonator sums before applying (6.4)–(6.5). The same bound, with $\mathfrak{u} = (1)$, applies to all products of two or four resonator ideals in S_0 and the fourth moment. No assertion of uniformity over the untruncated $\mathfrak{k}$-sum is needed: its tail was bounded absolutely above before applying character orthogonality. Terms divisible by a prime in S vanish by (2.7). Equations (6.4) and (6.5) therefore bound the combined character-average error by

$$O_{K,\Omega,W,A_0,\varepsilon}\left(X^{3/4+\varepsilon+o(1)}N^2\right) = o(XN). \tag{6.6}$$

The last equality uses $\varepsilon < 1/4 - \beta$.

Both square-diagonal main terms are nonnegative. This follows from the positivity of the kernels, the dual factor (4.3), and h_K. In the first main term, retain, for each pair $\mathfrak{m}, \mathfrak{n} \in \mathcal{M}$, only

$$\mathfrak{k} = \frac{[\mathfrak{m},\mathfrak{n}]}{(\mathfrak{m},\mathfrak{n})}.$$

This choice makes $\mathfrak{k}\mathfrak{m}\mathfrak{n} = [\mathfrak{m},\mathfrak{n}]^2$ and gives the weight

$$(\mathrm{N}\mathfrak{k})^{-s} = \left(\frac{\mathrm{N}(\mathfrak{m},\mathfrak{n})}{\mathrm{N}[\mathfrak{m},\mathfrak{n}]}\right)^{1/2+\delta}.$$

It is therefore exactly the kernel in the moving Gál sum. The support of $\mathfrak{k}\mathfrak{m}\mathfrak{n}$ lies among primes of norm at most $y_{\mathcal{M}}$. The prime ideal theorem and partial summation give $\sum_{\mathrm{N}\mathfrak{p} \le y}(\mathrm{N}\mathfrak{p})^{-1} \le \log\log y + O_K(1)$ for $y \ge 3$. Using $\log(1+u) \le u$, we obtain

$$h_K(\mathfrak{k}\mathfrak{m}\mathfrak{n}) \ge \prod_{\substack{\mathrm{N}\mathfrak{p} \le y_{\mathcal{M}} \\ \mathfrak{p} \notin S}} (1+(\mathrm{N}\mathfrak{p})^{-1})^{-1} \ge \exp\{-O_K(\log_3 N)\} = \exp\{-o(\mathcal{V}(N))\}. \tag{6.7}$$

Retain only pairs for which $\mathrm{N}\mathfrak{k} \le X^\varepsilon$. Since $\varepsilon < 1/2$, (4.4) gives

$$\int_X^{2X} w^+_{\mathfrak{k},X,A}(t)\,dt = XI_W\left(1+O\left(X^{-\rho(1/2-\varepsilon)}\right)\right)$$

for any fixed $0 < \rho < 1/2 - \delta_0$. Lemma 5.5 supplies the sum of their weights. Since $\mathcal{V}(N) = (\sqrt{\beta} + o(1))\mathcal{V}(X)$, we conclude that

$$S_1(A) \geq XN \exp\{(2e^{-A}\sqrt{\beta} + o(1))\mathcal{V}(X)\}. \tag{6.8}$$

At $A = 0$, both approximate-functional-equation sums are equal. Keeping both gives the more precise central estimate

$$S_1 \geq 2\kappa_\Omega I_W XN \exp\{(2 + o(1))\mathcal{V}(N)\}. \tag{6.9}$$

All error terms above are uniform for $0 \leq A \leq A_0$.

The lower bound for $S_1(A)$ is positive, so $S_0 > 0$. Its quotient by S_0 is a weighted average of the real values $L(s, \chi_\mathfrak{a})$. Hence

$$\max_{\mathfrak{a} \in \Omega(X)} L(s, \chi_\mathfrak{a}) \geq \frac{S_1(A)}{S_0} \geq \exp\{(2e^{-A}\sqrt{\beta} + o(1))\mathcal{V}(X)\}.$$

For every fixed $\beta < 1/4$, this holds uniformly on $[0, A_0]$. Given any $\tau > 0$, first take β close enough to $1/4$ that $2\sqrt{\beta} > 1 - \tau$, and then take X sufficiently large. The following choice realizes the displayed $o(1)$ as a single function of X. If one takes a sequence $\beta_j \uparrow 1/4$ and chooses increasing thresholds on which the estimate for β_j is uniform in $0 \leq A \leq A_0$, the resulting piecewise-constant choice gives a coefficient $e^{-A} + o(1)$ uniformly on the compact A-interval. This proves the maximum assertion, including its uniformity.

For abundance, fix $0 < c < 1$ and choose

$$\frac{c^2}{4} < \beta < \frac{1}{4}. \tag{6.10}$$

Let

$$\mathcal{H}_{A,c}(X) := \{\mathfrak{a} \in \Omega(X) : L(s, \chi_\mathfrak{a}) \geq e^{ce^{-A}\mathcal{V}(X)}\}.$$

Outside this set, the first moment contribution is at most $e^{ce^{-A}\mathcal{V}(X)}S_0$. This upper bound also holds for negative L-values. By (6.3), we have $S_0 \leq CXN$ with a fixed constant C. The discarded contribution divided by the lower bound in (6.8) is therefore at most

$$C \exp\{-[e^{-A}(2\sqrt{\beta} - c) + o(1)]\mathcal{V}(X)\} = o(1).$$

The gap is bounded below by $e^{-A_0}(2\sqrt{\beta} - c) > 0$, so this estimate is uniform in A. It follows that

$$\sum_{\mathfrak{a} \in \mathcal{H}_{A,c}(X)} W(\mathrm{N}\mathfrak{a}/X) L(s, \chi_\mathfrak{a}) R_\mathfrak{a}^2 \geq XN \exp\{(2e^{-A}\sqrt{\beta} + o(1))\mathcal{V}(X)\}. \tag{6.11}$$

Put $U_K(X) := \exp\{C_K \log X / \log_2 X\}$, as in Lemma 4.2. The L-values on $\mathcal{H}_{A,c}(X)$ are positive and at most $U_K(X)$. Cauchy–Schwarz and Lemma 6.1 give

$$\begin{aligned}
&\sum_{\mathfrak{a} \in \mathcal{H}_{A,c}(X)} W(\mathrm{N}\mathfrak{a}/X) L(s, \chi_\mathfrak{a}) R_\mathfrak{a}^2 \\
&\leq U_K(X) \left(\sum_{\mathfrak{a} \in \mathcal{H}_{A,c}(X)} W(\mathrm{N}\mathfrak{a}/X) \right)^{1/2} \left(\sum_{\mathfrak{a} \in \Omega} W(\mathrm{N}\mathfrak{a}/X) |R_\mathfrak{a}|^4 \right)^{1/2} \\
&\ll U_K(X)\, \#\mathcal{H}_{A,c}(X)^{1/2} X^{1/2} N X^{o(1)}.
\end{aligned}$$

Comparing this with (6.11) and squaring yields

$$\#\mathcal{H}_{A,c}(X) \geq X U_K(X)^{-2} X^{-o(1)} \exp\{(4e^{-A}\sqrt{\beta} + o(1))\mathcal{V}(X)\} = X^{1-o(1)}.$$

Here both $\log U_K(X)$ and $\mathcal{V}(X)$ are $o(\log X)$. In particular

$$\log\bigl(U_K(X)^{-2} \exp\{O(\mathcal{V}(X))\}\bigr) = o(\log X),$$

which justifies absorbing all remaining factors into $X^{o(1)}$. This proves the second assertion, uniformly on $[0, A_0]$.

Finally, we choose one threshold approaching the endpoint. For $j \geq 1$, put

$$c_j := 1 - \frac{3}{10j}, \qquad \beta_j := \frac{1}{4} - \frac{1}{10j}.$$

Writing $u = 1/(10j)$, we have $c_j^2 - 4\beta_j = u(9u-2) < 0$. The fixed-parameter abundance estimate therefore applies to (c_j, β_j). Choose an increasing sequence $X_j \to \infty$ such that, for all $X \geq X_j$ and $0 \leq A \leq A_0$, it gives $\#\mathcal{H}_{A,c_j}(X) \geq X^{1-1/j}$. For $X_j \leq X < X_{j+1}$, define

$$\eta_{A_0}(X) := 1 - c_j = \frac{3}{10j}.$$

Then $\eta_{A_0}(X) \to 0$, and the asserted count is at least $X^{1-1/j} = X^{1-o(1)}$ uniformly on $[0, A_0]$. Notice that this argument produces one sufficiently slowly decreasing function η_{A_0}. It does not assert the conclusion for an arbitrary prescribed function $\eta(X) \to 0$, nor does it give an effective decay rate. Taking $A_0 = 0$ gives the central endpoint statement with $\eta(X) = \eta_0(X)$. □

7. Global function fields

Let $C/\mathbb{F}_q$ be a fixed smooth, projective, geometrically connected curve, where q is odd, and put $F = \mathbb{F}_q(C)$. Write g_C for its genus. All constants in this section may depend on this curve and on the fixed place chosen below. The fixed local condition determines the family in which we prove the prime average. With this estimate, the resonance argument uses the finite block calculation from Section 5.

7.1. The family and its prime averages.

Choose a place ∞ of odd degree d_∞. Such a place exists by the prime-divisor theorem [32, Theorem 5.12]. Put

$$\mathscr{A} = \Gamma(C \setminus \{\infty\}, \mathcal{O}_C), \qquad S = \{\infty\}, \qquad h_{\mathscr{A}} = |\operatorname{Cl}(\mathscr{A})|.$$

The ring $\mathscr{A}$ is Dedekind, and its units are $\mathbb{F}_q^\times$. We identify its prime ideals with the places different from ∞. If $\mathfrak{a} = \prod_v \mathfrak{p}_v^{e_v}$ is a nonzero integral ideal, define $\deg \mathfrak{a} = \sum_v e_v \deg v$ and $|\mathfrak{a}| = q^{\deg \mathfrak{a}}$. Thus sums over ideals are sums over effective divisors away from ∞.

Fix a uniformizer ϖ_∞, and define the leading coefficient

$$\operatorname{sgn}_\infty(f) = \overline{f \varpi_\infty^{-v_\infty(f)}} \in \mathbb{F}_{q^{d_\infty}}^\times \qquad (f \in F^\times).$$

Since d_∞ is odd, a constant is a square in this residue field if and only if it is a square in $\mathbb{F}_q$. For degrees

$$n = d_\infty(2r+1), \qquad r \geq 0, \tag{7.1}$$

let $\Omega_F(n)$ be the principal prime ideals of $\mathscr{A}$ of degree n. For $\mathfrak{P} \in \Omega_F(n)$, choose a generator $\pi_\mathfrak{P}$ with

$$(\pi_\mathfrak{P}) = \mathfrak{P} - \frac{n}{d_\infty}\infty, \qquad \operatorname{sgn}_\infty(\pi_\mathfrak{P}) \in \mathbb{F}_{q^{d_\infty}}^{\times 2}. \tag{7.2}$$

Multiplication by a nonsquare constant changes the square class of the leading coefficient. Two generators satisfying this condition differ by a square constant. Hence the quadratic extension $F(\sqrt{\pi_\mathfrak{P}})/F$ is well defined. Let $\chi_\mathfrak{P}$ be its nontrivial Artin character. Its coefficients at ramified places are defined to be zero.

This normalization serves two purposes. It fixes the ramification at ∞, and it makes evaluation at a fixed ideal into a ray class character of the varying prime. The latter property gives an error depending only linearly on the degree of the fixed ideal.

The following prime estimate for geometrically nontrivial characters gives the family count without using the later orthogonality proposition.

Lemma 7.1. *Let θ be a geometrically nontrivial finite-order idele class character of F. Then, uniformly for $n \geq 1$,*

$$\sum_{\substack{\deg \mathfrak{P}=n\\ \mathfrak{P}\neq\infty}} \theta(\mathfrak{P}) \ll_F \frac{(1+\deg \mathfrak{f}_\theta)q^{n/2}}{n}. \tag{7.3}$$

Proof. The complete L-function of θ is a polynomial of degree $D_\theta = 2g_C - 2 + \deg \mathfrak{f}_\theta$. See [32, Theorem 9.24A, p. 141]. To invoke Weil's theorem in a manifestly geometric form, choose a degree-one divisor D whose support is disjoint from the conductor and put $\widetilde{\theta} = \theta\,\theta(D)^{-\deg}$. We do not require a degree-one place: by the prime-divisor theorem, for all sufficiently large m there are places P_m and P_{m+1} of degrees m and $m+1$, and they may be chosen outside the finite conductor. Then $D = P_{m+1} - P_m$ has degree one. The twisted character agrees with θ on degree-zero idele classes and satisfies $\widetilde{\theta}(D) = 1$. Every idele class of degree m is the product of a degree-zero class and D^m, so the image of $\widetilde{\theta}$ equals its image on the degree-zero subgroup. Thus the cyclic extension associated with $\widetilde{\theta}$ has no nontrivial constant-field quotient and is geometric. See [32, Proposition 9.22]. Its character remains nontrivial. This conclusion concerns the twisted character: geometric nontriviality alone need not make the original cyclic extension geometric before removal of its degree component. Therefore Weil's theorem [32, Theorem 9.16B, p. 129] gives inverse roots of $L(u,\widetilde{\theta})$ of absolute value $q^{1/2}$. Undoing the degree twist only rotates these roots, so the inverse roots $\alpha_{\theta,j}$ of $L(u,\theta)$ have the same absolute value.

Logarithmic differentiation gives

$$\sum_{\deg \mathfrak{d}=n} \Lambda_F(\mathfrak{d})\theta(\mathfrak{d}) = -\sum_{j=1}^{D_\theta} \alpha_{\theta,j}^n.$$

The left side runs over all effective divisors on C, with $\Lambda_F(v^k) = \deg v$ on prime powers and zero otherwise. Removing prime powers of exponent at least two costs $O_F(q^{n/2})$: prime squares contribute $O_F(q^{n/2})$, while higher powers contribute $O_F(nq^{n/3}) = O_F(q^{n/2})$. Removing the fixed place ∞ costs only $O_F(1)$ (its von Mangoldt weight is d_∞ when it occurs). Division by n proves (7.3). ☐

Proposition 7.2. *For the degrees in (7.1), the following hold.*

(i) *Each $\chi_{\mathfrak{P}}$ has conductor*

$$\mathfrak{f}_{\mathfrak{P}} = \mathfrak{P} + \infty. \tag{7.4}$$

In particular, its coefficient at ∞ is zero.

(ii) *With $\kappa_F = d_\infty/h_{\mathscr{A}} > 0$, one has*

$$\#\Omega_F(n) = \kappa_F \frac{q^n}{n} + O_F\left(\frac{q^{n/2}}{n}\right). \tag{7.5}$$

Proof. The valuations of $\pi_{\mathfrak{P}}$ are odd at $\mathfrak{P}$ and ∞, and zero elsewhere. Quadratic Kummer ramification is tame because q is odd. The conductor exponent is one at these two places and zero elsewhere. This proves the conductor formula.

Extend each character ξ of $\mathrm{Cl}(\mathscr{A})$ to divisors on C by setting $\xi(\infty) = 1$. The extension is trivial on principal divisors, and so is an unramified idele class character. Exactly d_∞ of these characters factor through the degree map: they are $\xi(\mathfrak{d}) = \zeta^{\deg \mathfrak{d}}$ with $\zeta^{d_\infty} = 1$. Indeed, the prime-divisor theorem supplies places of two consecutive sufficiently large degrees, whose difference is a divisor of degree one. Thus the degree map on divisor classes is onto $\mathbb{Z}$, and after quotienting by the class of ∞ the degree map on $\mathrm{Cl}(\mathscr{A})$ has image $\mathbb{Z}/d_\infty\mathbb{Z}$. Every other character is geometrically nontrivial and satisfies Lemma 7.1 with conductor zero. Since $d_\infty \mid n$, each of the d_∞ degree characters takes the value one on every prime of degree n.

Character orthogonality and [32, Theorem 5.12] now give

$$\#\Omega_F(n) = \frac{1}{h_{\mathscr{A}}} \sum_{\xi} \sum_{\substack{\deg \mathfrak{P}=n \\ \mathfrak{P}\neq\infty}} \xi(\mathfrak{P})$$
$$= \frac{d_\infty}{h_{\mathscr{A}}} \left(\frac{q^n}{n} + O_F(q^{n/2}/n) \right) + O_F(q^{n/2}/n).$$

This proves (7.5). □

Proposition 7.3. *Let $\mathfrak{l}$ be a nonzero integral ideal of $\mathscr{A}$ having no prime factor of degree n. Write $\mathfrak{l} = \mathfrak{l}_0\mathfrak{l}_1^2$, with $\mathfrak{l}_0$ squarefree. Uniformly in $\mathfrak{l}$,*

$$\sum_{\mathfrak{P}\in\Omega_F(n)} \chi_{\mathfrak{P}}(\mathfrak{l}) = \kappa_F \frac{q^n}{n} \mathbf{1}_{\mathfrak{l}_0=\mathscr{A}} + O_F\left(\frac{(1+\deg \mathfrak{l}_0)q^{n/2}}{n} \right). \tag{7.6}$$

Proof. We construct a reciprocal character for $\mathfrak{l}_0$. Let $I_{\mathscr{A}}(\mathfrak{l}_0)$ be the fractional ideals prime to $\mathfrak{l}_0$, and let $P^+_{\mathscr{A}}(\mathfrak{l}_0)$ consist of the principal ideals $(a)_{\mathscr{A}}$ satisfying

$$a \equiv 1 \pmod{\mathfrak{l}_0}, \qquad \mathrm{sgn}_\infty(a) \in \mathbb{F}_{q^{d_\infty}}^{\times 2}.$$

The congruence requires a to be a unit at every prime of the modulus. The quotient

$$G(\mathfrak{l}_0) = I_{\mathscr{A}}(\mathfrak{l}_0)/P^+_{\mathscr{A}}(\mathfrak{l}_0) \tag{7.7}$$

is finite. Indeed, it maps onto $\mathrm{Cl}(\mathscr{A})$, and its kernel is a quotient of

$$\prod_{v\mid\mathfrak{l}_0} \mathbb{F}^\times_{q^{\deg v}} \times \left(\mathbb{F}^\times_{q^{d_\infty}} / \mathbb{F}^{\times 2}_{q^{d_\infty}} \right).$$

Let $H(\mathfrak{l}_0)$ be the subgroup represented by principal ideals. For $I = (a)_{\mathscr{A}}$, multiply a by a constant to make its leading coefficient square, and denote the result by $a^\sharp$. Define

$$\lambda_{\mathfrak{l}_0}(I) = \prod_{v\mid\mathfrak{l}_0} \left(\frac{\overline{a^\sharp}_v}{v} \right)_2. \tag{7.8}$$

Here $(\,\cdot\,/v)_2$ is the quadratic character of the residue field. The choice of $a^\sharp$ is unique up to a square constant. Thus the formula is well defined and multiplicative. It is trivial on $P^+_{\mathscr{A}}(\mathfrak{l}_0)$.

A character of a subgroup of a finite abelian group extends to the group. Choose an extension $\rho_{\mathfrak{l}_0}$ of $\lambda_{\mathfrak{l}_0}$ to $G(\mathfrak{l}_0)$. As an idele class character, it has conductor dividing $\mathfrak{l}_0 + \infty$. The local Artin symbol and the degree assumption give

$$\chi_{\mathfrak{P}}(\mathfrak{l}) = \prod_{v\mid\mathfrak{l}_0} \left(\frac{\overline{\pi_{\mathfrak{P}}}_v}{v} \right)_2 = \rho_{\mathfrak{l}_0}(\mathfrak{P}) \qquad (\mathfrak{P}\in\Omega_F(n)). \tag{7.9}$$

Inflate the class group characters to $G(\mathfrak{l}_0)$. Their orthogonality yields

$$\sum_{\mathfrak{P}\in\Omega_F(n)} \chi_{\mathfrak{P}}(\mathfrak{l}) = \frac{1}{h_{\mathscr{A}}} \sum_{\xi\in\widehat{\mathrm{Cl}(\mathscr{A})}} \sum_{\substack{\deg \mathfrak{P}=n \\ \mathfrak{P}\neq\infty}} (\rho_{\mathfrak{l}_0}\xi)(\mathfrak{P}). \tag{7.10}$$

Suppose $\mathfrak{l}_0 \neq \mathscr{A}$, and fix $v_0 \mid \mathfrak{l}_0$. Weak approximation gives $a \in F^\times$ with residue 1 at ∞ and at every $v \mid \mathfrak{l}_0$ other than v_0, and a nonsquare residue at v_0. Then $\lambda_{\mathfrak{l}_0}((a)_{\mathscr{A}}) = -1$. Also, $\deg(a)_{\mathscr{A}} = 0$ because $v_\infty(a) = 0$. Since ξ is inflated from the ideal class group, it is trivial on the principal ideal $(a)_{\mathscr{A}}$, whereas the chosen extension satisfies $\rho_{\mathfrak{l}_0}((a)_{\mathscr{A}}) = \lambda_{\mathfrak{l}_0}((a)_{\mathscr{A}}) = -1$. Hence

$$(\rho_{\mathfrak{l}_0}\xi)((a)_{\mathscr{A}}) = -1.$$

The class of $(a)_{\mathscr{A}}$ has degree zero, so every character $\theta = \rho_{\mathfrak{l}_0}\xi$ in (7.10) is nontrivial on the degree-zero idele classes. This is precisely geometric nontriviality. In particular, none of these characters factors through the degree map. Moreover,

$$\deg \mathfrak{f}_\theta \le \deg \mathfrak{l}_0 + d_\infty.$$

Applying Lemma 7.1 to each $\theta = \rho_{\mathfrak{l}_0}\xi$ and using $\deg \mathfrak{f}_\theta \le \deg \mathfrak{l}_0 + d_\infty$ in (7.10) proves the required estimate for nonsquares. If $\mathfrak{l}_0 = \mathscr{A}$, then $\mathfrak{l}$ is a square and, by the hypothesis excluding degree-n prime factors, $\chi_{\mathfrak{P}}(\mathfrak{l}) = 1$ for every $\mathfrak{P} \in \Omega_F(n)$. Hence the left side of (7.6) is exactly $\#\Omega_F(n)$, and (7.5) proves the assertion. □

7.2. **The functional equation and positivity.** Put $d_n = n + d_\infty + 2g_C - 2$. This is even on the degrees in (7.1). Write $L(u,\chi)$ for the polynomial variable, so that $L(s,\chi) = L(q^{-s},\chi)$ in the complex variable s.

Lemma 7.4. *For $\mathfrak{P} \in \Omega_F(n)$, the following hold.*

(i) *The polynomial has degree d_n and satisfies*

$$L(u,\chi_{\mathfrak{P}}) = (q^{1/2}u)^{d_n} L((qu)^{-1},\chi_{\mathfrak{P}}). \tag{7.11}$$

(ii) *For every $\delta \ge 0$,*

$$L(1/2+\delta,\chi_{\mathfrak{P}}) = \sum_{\mathfrak{a}} \frac{\chi_{\mathfrak{P}}(\mathfrak{a})}{|\mathfrak{a}|^{1/2+\delta}} V_{d_n,\delta}(\deg \mathfrak{a}), \tag{7.12}$$

where the sum is over nonzero integral ideals of $\mathscr{A}$, and

$$V_{d,\delta}(r) = \begin{cases} 1 + q^{-\delta(d-2r)}, & 2r < d, \\ 1, & 2r = d, \\ 0, & 2r > d. \end{cases} \tag{7.13}$$

In particular, this weight lies in $[1,2]$ on $2r \le d$.

(iii) *One has $L(\sigma,\chi_{\mathfrak{P}}) \ge 0$ for real $\sigma \ge 1/2$, with strict positivity for $\sigma > 1/2$.*

Proof. Let $C_{\mathfrak{P}}$ be the smooth projective curve with function field $F(\sqrt{\pi_{\mathfrak{P}}})$. Ramification shows that the extension has full constant field $\mathbb{F}_q$. Artin factorization gives

$$Z_{C_{\mathfrak{P}}}(u) = Z_C(u) L(u,\chi_{\mathfrak{P}}). \tag{7.14}$$

The two curve zeta functions have the same denominator $(1-u)(1-qu)$. Their numerator polynomials satisfy functional equations with sign $+1$. Riemann–Hurwitz gives

$$2g_{C_{\mathfrak{P}}} - 2 = 2(2g_C - 2) + n + d_\infty, \qquad 2(g_{C_{\mathfrak{P}}} - g_C) = d_n.$$

Taking the quotient of the two functional equations proves (7.11).

To obtain the finite sum, write $L(u,\chi_{\mathfrak{P}}) = \sum_{r=0}^{d_n} a_r u^r$. Its coefficients are $a_r = \sum_{\deg \mathfrak{a} = r} \chi_{\mathfrak{P}}(\mathfrak{a})$, since the factor at ∞ is one. Comparing coefficients in the functional equation gives $a_{d_n - r} = q^{d_n/2 - r} a_r$. At $u = q^{-1/2-\delta}$, a pair of terms with $2r < d_n$ is therefore

$$a_r q^{-(1/2+\delta)r} + a_{d_n - r} q^{-(1/2+\delta)(d_n - r)} = a_r q^{-(1/2+\delta)r}\{1 + q^{-\delta(d_n - 2r)}\}.$$

The middle term occurs once. This proves the formula. At $\delta = 0$, it is the usual exact central formula. Compare [4, Section 3.5, Lemma 1].

By Weil's theorem,

$$L(q^{-1/2}z,\chi_{\mathfrak{P}}) = \prod_{j=1}^{d_n} (1 - z e^{i\theta_j}).$$

The coefficients are real. A conjugate pair contributes $|1 - z e^{i\theta_j}|^2$, while a real factor is $1 - z$ or $1 + z$. All these factors are nonnegative for $0 \le z \le 1$, and positive for $0 \le z < 1$. Take $z = q^{1/2-\sigma}$. □

For comparison with polynomial families, take $F = \mathbb{F}_q(t)$ and $\deg\infty = 1$. Our convention is $\chi_P(f) = (P/f)$, whereas [11] uses (f/P) for monic polynomials. For odd $\deg P = n$, quadratic reciprocity gives

$$(P/f) = (-1)^{((q-1)/2)\deg f}(f/P).$$

The conventions agree when $q \equiv 1 \pmod 4$. When $q \equiv 3 \pmod 4$, set $P^*(t) = -P(-t)$. This is an involution on the monic primes of odd degree. For $\deg f = r$, the substitution $f^*(t) = (-1)^r f(-t)$ permutes the monic polynomials of degree r and gives

$$(f^*/P^*) = (-1)^r(f/P).$$

Thus the coefficient sums for our character at P equal those for the convention in [11] at P^*. The two conventions give the same multiset of L-values on the family, so the comparison of lower-bound constants is unaffected.

7.3. **A Gál sum with prescribed prime degrees.** The prime-divisor theorem has the form

$$\#\{\mathfrak{p} : \deg\mathfrak{p} = j\} = \frac{q^j}{j} + O_F(q^{j/2}/j) \tag{7.15}$$

by [32, Theorem 5.12]. We use one exact degree per block. All weights in a block are then equal, which avoids any loss from the discreteness of the possible norms.

Proposition 7.5. *For every sufficiently large integer N, there is a set $\mathcal{M}_F$ of N squarefree integral ideals of $\mathscr{A}$ with the following properties.*

(i) *Uniformly for A in a fixed compact subset of $[0,\infty)$,*

$$\sum_{\mathfrak{m},\mathfrak{r}\in\mathcal{M}_F}\left(\frac{|(\mathfrak{m},\mathfrak{r})|}{|[\mathfrak{m},\mathfrak{r}]|}\right)^{1/2+A/\log_2 N} \geq N\exp\{(2e^{-A}+o(1))\mathcal{V}(N)\}. \tag{7.16}$$

(ii) *Its prime factors have norm at most $(\log N)^{1+o(1)}$, and*

$$\max_{\mathfrak{m}\in\mathcal{M}_F}\deg\mathfrak{m} \ll_F \frac{\log N\log_2 N}{\log_3 N}, \qquad E_\Box(\mathcal{M}_F) \leq N^{2+o(1)}. \tag{7.17}$$

(iii) *If $N \leq q^{n/4}$, restricting the sum to*

$$\deg\frac{[\mathfrak{m},\mathfrak{r}]}{(\mathfrak{m},\mathfrak{r})} \leq \vartheta n \tag{7.18}$$

for fixed $\vartheta > 0$ preserves the lower bound. If $N = X_n^{\beta+o(1)}$ with fixed $\beta > 0$, the exponent $A/\log_2 N$ may also be replaced by $A/\log_2 X_n$.

Proof. Fix $0 < \gamma < 1$ and $0 < \eta < 1/2$ for now, and abbreviate $T_i = \log_i N$. Set

$$D_N = \lceil\log_q(T_1T_2)\rceil, \qquad K_N = \lfloor T_2^\gamma\rfloor.$$

Let $\mathscr{P}_k$ be the primes of $\mathscr{A}$ of degree $D_N + k$, where $1 \leq k \leq K_N$, and put $P_k = |\mathscr{P}_k|$. Since $K_N = o(D_N)$, (7.15) gives, uniformly over the blocks,

$$P_k = c_N q^k T_1(1+o(1)), \quad c_N \asymp_q 1, \qquad P_k q^{-(D_N+k)} = \frac{\log q + o(1)}{T_2}. \tag{7.19}$$

For fixed $a > 0$, take all products containing at most $J_k(a) = \lfloor aT_1/(k^2T_3)\rfloor$ primes in block k. Denote the product of these families by $\mathcal{M}_F(a)$. The binomial estimate (5.15) gives

$$\log|\mathcal{M}_F(a)| = (a\gamma\log q + o(1))T_1. \tag{7.20}$$

Here is the calculation of its leading term. Uniformly in k,

$$\log(P_k/J_k(a)) = k\log q + 2\log k + T_4 + O_{q,a}(1), \qquad J_k(a)/P_k = o(1).$$

The first term contributes $\log q \sum_k kJ_k(a) = (a\gamma \log q + o(1))T_1$. The remaining terms are bounded by

$$O_{q,a}\left(\frac{T_1(1+T_4)}{T_3} + K_N T_2 + \frac{T_1}{T_3^2}\right) = o(T_1).$$

This includes the floor errors and the terms $\sum_k J_k(a)^2/P_k \ll_q T_1/T_3^2$.

Put $a_\pm = (1 \pm \eta)/(\gamma \log q)$. Starting with the quotas $J_k(a_-)$, increase them one at a time toward $J_k(a_+)$. Stop immediately before the size first exceeds N. The resulting product family $\mathcal{M}_*$ has size $M \le N$ and quotas J_k between these two bounds. Comparing consecutive binomial tails gives

$$N/M \le 1 + \max_k P_k \le T_1^{1+o(1)}. \tag{7.21}$$

We apply Lemma 5.2 to the shifted weights

$$w_k(A) = q^{-(1/2+A/T_2)(D_N+k)}, \qquad H_k(A) = P_k w_k(A).$$

Since $(D_N + k)\log q = T_2 + T_3 + k \log q + O_q(1)$, one has

$$w_k(A) = e^{-A+o(1)} q^{-(D_N+k)/2} \tag{7.22}$$

uniformly in k and in bounded A. Choose

$$r_k(A) = \left\lfloor \frac{e^{-A}\sqrt{a_- \log q}}{k} \sqrt{\frac{T_1}{T_2 T_3}} \right\rfloor.$$

For A in a fixed compact interval, these integers tend to infinity uniformly in k. The remaining hypotheses of the block lemma follow from

$$\frac{r_k(A)}{J_k} \ll \frac{K_N\sqrt{T_3}}{\sqrt{T_1 T_2}} = o(1), \qquad \frac{J_k}{P_k} \ll_q T_3^{-1} = o(1),$$

and

$$\frac{J_k + r_k(A)^2}{P_k} \ll_q \frac{q^{-k}}{k^2 T_3} + \frac{q^{-k}}{k^2 T_2 T_3} = o(1).$$

Moreover, (7.19) implies

$$\frac{e^2 J_k H_k(A)^2}{P_k r_k(A)^2} \ge e^2(1 + o(1)).$$

Let $S_s(\mathcal{B})$ denote the Gál sum over $\mathcal{B}$ with exponent s. Factoring it over the blocks gives

$$\begin{aligned} \log \frac{S_{1/2+A/T_2}(\mathcal{M}_*)}{M} &\ge (2 + o(1)) \sum_k r_k(A) \\ &= \left(2e^{-A}\sqrt{\gamma(1-\eta)} + o(1)\right)\mathcal{V}(N). \end{aligned} \tag{7.23}$$

The factor $\log q$ in the block density cancels the factor $\log q$ in the cardinality estimate. This explains why the leading constant does not depend on q.

To obtain exactly N elements, put $h = \lfloor \log(N/M)/\log 2 \rfloor$. Choose h new primes, each of degree $O_F(\log(h+2))$, and take all products of these primes. Call this cube $\mathcal{C}_h$. Its support is disjoint from the original blocks, and

$$\frac{S_s(\mathcal{C}_h)}{|\mathcal{C}_h|} = \prod_{\mathfrak{q}\ \text{new}} (1 + |\mathfrak{q}|^{-s}) \ge 1.$$

Thus $\mathcal{M}_1 = \mathcal{C}_h \mathcal{M}_*$ has size in $(N/2, N]$, and

$$\frac{S_s(\mathcal{M}_1)}{|\mathcal{M}_1|} \ge \frac{S_s(\mathcal{M}_*)}{M}. \tag{7.24}$$

Add elements from $\mathcal{M}_F(a_+) \setminus \mathcal{M}_1$ until the size is N. There are enough, since $|\mathcal{M}_F(a_+)| > N$ and its intersection with $\mathcal{M}_1$ is $\mathcal{M}_*$. Positivity of the kernel shows that this completion costs a factor of at most two in (7.23).

The energy estimate survives the completion. Lemma 5.1 gives

$$\log \frac{E_\square(\mathcal{M}_F(a_+))}{|\mathcal{M}_F(a_+)|^2} \ll \sum_k \{J_k(a_+) + J_k(a_+)^2/P_k + \log(J_k(a_+)+1)\}$$
$$\ll_F T_1/T_3 + T_1/T_3^2 + K_N T_2 = o(T_1). \tag{7.25}$$

Every completed element lies in $\mathcal{C}_h\mathcal{M}_F(a_+)$. Its energy is $2^{3h}E_\square(\mathcal{M}_F(a_+)) \le N^{2+2\eta+o(1)}$, because $2^h \le T_1^{1+o(1)}$.

A diagonal choice of the parameters gives the limiting constant. For $j \ge 3$, take $\gamma_j = 1-1/j$ and $\eta_j = 1/j$. Choose increasing thresholds N_j so that all preceding errors are at most $1/j$ for $N \ge N_j$, uniformly for $0 \le A \le j$. Require also

$$E_\square(\mathcal{C}_h\mathcal{M}_F(a_+)) \le N^{2+5/j}, \qquad T_2^{-1/j} \le 1/j.$$

Use these parameters on $N_j \le N < N_{j+1}$. Then (7.23) proves (7.16), and the energy is $N^{2+o(1)}$. The extra threshold ensures $K_N/D_N = o(1)$ along this choice. Consequently every original prime has

$$\log|\mathfrak{p}| = (1+o(1))\log_2 N,$$

and the auxiliary primes have smaller degree. An element of the upper family has degree at most

$$\sum_k J_k(a_+)(D_N+k) \ll_F \frac{T_1}{T_3}\left(D_N\sum_{k\ge1}k^{-2} + \sum_{k\le K_N}k^{-1}\right) \ll_F \frac{T_1T_2}{T_3}.$$

The auxiliary primes add $O_F(T_2T_3)$. This proves all size assertions.

For the truncation, let $\mathscr{Q}_N$ contain all supporting primes. Its largest degree is $(1+o(1))\log_q\log N$. Put $E = \deg([\mathfrak{m},\mathfrak{r}]/(\mathfrak{m},\mathfrak{r}))$. When $E > \vartheta n$, we have

$$q^{-E/2} \le q^{-\vartheta n/6}q^{-E/3}.$$

For fixed $\mathfrak{m}$, enlarge the sum over $\mathfrak{r}$ to all squarefree products on $\mathscr{Q}_N$. The resulting bound is

$$q^{-\vartheta n/6}\prod_{\mathfrak{p}\in\mathscr{Q}_N}(1+|\mathfrak{p}|^{-1/3}) \le q^{-\vartheta n/6}\exp\{(\log N)^{2/3+o(1)}\}.$$

Summing over $\mathfrak{m}$ gives $o(N)$ when $N \le q^{n/4}$. The shifted kernel is at most the central kernel, so the same bound applies uniformly in $A \ge 0$. Finally, if $\log N = (\beta+o(1))\log X_n$, put $B = A\log_2 N/\log_2 X_n$. Then $A/\log_2 X_n = B/\log_2 N$ and $B = A + o(1)$ uniformly for bounded A. Applying the compact-uniform estimate already proved at B gives the stated replacement of the shift. $\square$

7.4. **A uniform upper bound.** The fourth moment controls the weights placed on a small exceptional set. To turn that control into a count of large values, we also need an upper bound for each L-value. The following subpower bound is sufficient.

Lemma 7.6. *Uniformly for $\mathfrak{P} \in \Omega_F(n)$ and real $\sigma \ge 1/2$,*

$$0 \le L(\sigma,\chi_{\mathfrak{P}}) \le \exp\{O_F(n\log_2 n/\log n)\} = X_n^{o(1)}. \tag{7.26}$$

Proof. Positivity was proved in Lemma 7.4. Write $Q(z) = L(q^{-1/2}z,\chi_{\mathfrak{P}}) = \prod_{j=1}^{d_n}(1-ze^{i\theta_j})$. For an integer $M \ge 3$, set $\rho = \exp(-\log M/M)$. The inequality

$$\frac{|1-e^{i\theta}|}{|1-\rho e^{i\theta}|} \le \frac{2}{1+\rho}$$

follows by maximizing $2(1-\cos\theta)/((1-\rho)^2+2\rho(1-\cos\theta))$. Expand $\log|1-\rho e^{i\theta}|$ into its absolutely convergent series. Its tail satisfies

$$\sum_{k>M}\frac{\rho^k}{k} \le \frac{\rho^{M+1}}{(M+1)(1-\rho)} \ll \frac{1}{M\log M}.$$

Since $\log(2/(1+\rho)) \ll \log M/M$, we obtain

$$\log|1-e^{i\theta}| \le -\sum_{k\le M} \frac{\rho^k \cos(k\theta)}{k} + O(\log M/M). \tag{7.27}$$

The inequality remains valid when the left side is $-\infty$.

Logarithmic differentiation of the Euler product gives

$$\sum_{j=1}^{d_n} e^{ik\theta_j} = -q^{-k/2} \sum_{\deg \mathfrak{a}=k} \Lambda_F(\mathfrak{a})\chi_{\mathfrak{P}}(\mathfrak{a}) \ll_F q^{k/2}.$$

The last estimate follows from (7.15). Apply (7.27) at $\theta_j+\phi$ and sum over j. Uniformly in real ϕ,

$$\log|Q(e^{i\phi})| \ll_F \frac{n\log M}{M} + \sum_{k\le M} \frac{q^{k/2}}{k} \ll_F \frac{n\log M}{M} + \frac{q^{M/2}}{M}.$$

Take $M = \lfloor \log_q n\rfloor$, adjusting the finitely many small values of n. The maximum-modulus principle extends this bound to $|z|\le 1$. Substitute $z = q^{1/2-\sigma}$ to obtain the assertion. □

7.5. **Proof of Theorem 1.3.**

Proof. We treat the center and the critical window together. Fix $A_0 \ge 0$, let $0\le A\le A_0$, and write $s = 1/2 + A/\log_2 X_n$. Fix $0<\beta<1/4$ and set $N = \lfloor X_n^\beta \rfloor$. Choose the set in Proposition 7.5, and define

$$R_{\mathfrak{P}} = \sum_{\mathfrak{m}\in\mathcal{M}_F} \chi_{\mathfrak{P}}(\mathfrak{m}), \quad T_0 = \sum_{\mathfrak{P}\in\Omega_F(n)} R_{\mathfrak{P}}^2, \quad T_4 = \sum_{\mathfrak{P}\in\Omega_F(n)} R_{\mathfrak{P}}^4,$$

and

$$T_{1,A} = \sum_{\mathfrak{P}\in\Omega_F(n)} L(s,\chi_{\mathfrak{P}}) R_{\mathfrak{P}}^2.$$

The supporting primes of the resonator have degree $O_F(\log n)$. Thus no twisting ideal formed from its elements has a prime factor of degree n, once n is large.

Since the resonator ideals are squarefree, their product is a square exactly when they are equal. Proposition 7.3 therefore gives

$$T_0 = \kappa_F \frac{X_n}{n} N + O_F(X_n^{1/2} N^2 n^{O(1)}) = (\kappa_F + o(1))\frac{X_n}{n} N. \tag{7.28}$$

For four resonator ideals, the square terms are counted by $E_\square(\mathcal{M}_F)$. Consequently

$$T_4 = \kappa_F \frac{X_n}{n} E_\square(\mathcal{M}_F) + O_F(X_n^{1/2} N^4 n^{O(1)}) \le \frac{X_n}{n} N^{2+o(1)}. \tag{7.29}$$

The relative error is $X_n^{-1/2+2\beta+o(1)} = o(1)$.

Insert (7.12) into $T_{1,A}$. There are $O_F(q^r)$ ideals of degree r, as follows from the rationality of the curve zeta function. Only $r \le d_n/2 = n/2 + O_F(1)$ occurs. For large n, these ideals also have no prime factor of degree n. The squarefree part of each twisting ideal has degree at most $r + 2\max_{\mathfrak{m}\in\mathcal{M}_F} \deg \mathfrak{m} = n^{O(1)}$. Since the weights are at most two and $s\ge 1/2$, the total orthogonality error is

$$\begin{aligned} &\ll_F X_n^{1/2} N^2 n^{O(1)} \sum_{r\le d_n/2} q^{r/2} \\ &\ll_F X_n^{3/4} N^2 n^{O(1)} = o(X_n N/n). \end{aligned} \tag{7.30}$$

This uses $\beta < 1/4$.

For each pair $\mathfrak{m},\mathfrak{r}$, retain the ideal $\mathfrak{a} = [\mathfrak{m},\mathfrak{r}]/(\mathfrak{m},\mathfrak{r})$. It satisfies $\mathfrak{a}\mathfrak{m}\mathfrak{r} = [\mathfrak{m},\mathfrak{r}]^2$. Restrict to $\deg\mathfrak{a} \le \vartheta n$ with fixed $0<\vartheta<1/2$. The polynomial weight is then at least one. Every other square main term is also nonnegative, so it may be omitted. The truncated Gál bound now gives

$$T_{1,A} \geq \frac{X_n}{n} N \exp\{(2e^{-A}\sqrt{\beta} + o(1))\mathcal{V}_F(n)\}. \tag{7.31}$$

The fixed factor κ_F is absorbed in the exponent error. All estimates are uniform for $0 \leq A \leq A_0$. Dividing by (7.28) proves the maximum bound with coefficient $2e^{-A}\sqrt{\beta}$. Letting $\beta \uparrow 1/4$ through a fixed sequence and choosing admissible degree thresholds diagonally gives the coefficient $e^{-A} + o(1)$ uniformly for $0 \leq A \leq A_0$.

To count large values, fix $0 < c < 1$ and choose $c^2/4 < \beta < 1/4$. Let

$$\mathcal{H}_{F,A,c}(n) = \{\mathfrak{P} \in \Omega_F(n) : L(s, \chi_{\mathfrak{P}}) \geq \exp(ce^{-A}\mathcal{V}_F(n))\}.$$

The complement contributes at most $\exp(ce^{-A}\mathcal{V}_F(n))T_0$ to $T_{1,A}$. By (7.28) and (7.31), this is $o(T_{1,A})$, uniformly in A, since $2\sqrt{\beta} - c > 0$. Thus Cauchy–Schwarz and Lemma 7.6 give

$$(1 - o(1))T_{1,A} \leq X_n^{o(1)} \#\mathcal{H}_{F,A,c}(n)^{1/2} T_4^{1/2}.$$

It follows that

$$\#\mathcal{H}_{F,A,c}(n) \geq \frac{T_{1,A}^2}{X_n^{o(1)} T_4} \geq \frac{X_n}{n} X_n^{-o(1)} = X_n^{1-o(1)}.$$

Finally, take $\beta_j = 1/4 - 1/(10j)$ and $c_j = 1 - 3/(10j)$. Then $c_j^2 < 4\beta_j$. For each fixed j, the preceding estimates hold beyond an admissible degree n_j, uniformly for $A \in [0, A_0]$. Increase the thresholds so that the large-value count is at least $X_n^{1-1/j}$ whenever $n \geq n_j$. Use the jth choice for $n_j \leq n < n_{j+1}$ and put $\eta_{F,A_0}(n) = 1 - c_j$ there. This gives the endpoint count and hence the endpoint maximum. Taking $A_0 = 0$ gives the central assertions. □

8. Large values in the open strip

For Theorem 1.4, we use a finite Euler product to favour characters which take the value 1 at small primes. The same choice of coefficients gives the quadratic Dirichlet results in [10, Theorems 4 and 5]. For related resonance arguments, see [26, 3]. We write $\pi_K(y)$ for the number of prime ideals of norm at most y.

Proof of Theorem 1.4. Fix $1/2 < \sigma < 1$ and $0 < b < 1$. Choose $a > 0$, to be specified below, and put $Y = a \log X \log_2 X$. Define a completely multiplicative function r by

$$r_{\mathfrak{p}} = b \quad (\mathrm{N}\mathfrak{p} \leq Y,\ \mathfrak{p} \notin S), \qquad r_{\mathfrak{p}} = 0 \quad \text{otherwise}.$$

For each member of the family, put

$$R_{\mathfrak{a}} = \prod_{\substack{\mathrm{N}\mathfrak{p} \leq Y \\ \mathfrak{p} \notin S}} (1 - b\chi_{\mathfrak{a}}(\mathfrak{p}))^{-1} = \sum_{\mathfrak{m}} r_{\mathfrak{m}} \chi_{\mathfrak{a}}(\mathfrak{m}).$$

The series converges absolutely, since it is a product of finitely many convergent geometric series. Fix a nonzero smooth function W, with $0 \leq W \leq 1$, supported in $(1, 2)$, and write $I_W = \int W(t)\, dt$.

Let $y = (\log X)^B$, with B as in Lemma 4.3. We may enlarge B if necessary and assume $B > 1$, so that $Y < y$ for all sufficiently large X. Put

$$P_\sigma(\chi) = \sum_{\mathrm{N}\mathfrak{p} \leq y} \frac{\chi(\mathfrak{p})}{(\mathrm{N}\mathfrak{p})^\sigma}, \qquad S_2 = \sum_{\mathfrak{a} \in \Omega} W(\mathrm{N}\mathfrak{a}/X) R_{\mathfrak{a}}^2.$$

We will show that the average of P_σ with weight WR^2 is large. Lemma 4.3 then gives the same lower bound for $\log L$, up to a bounded error. GRH and positivity for real $s > 1$ imply that $L(s, \chi_{\mathfrak{a}}) > 0$ for $1/2 < s \leq 1$, so this logarithm is real.

The square terms in the expansion of R^2 can be computed prime by prime. The two relevant geometric sums are

$$D_b = \sum_{\substack{i,j\ge 0\\ i+j \text{ even}}} b^{i+j} = \frac{1+b^2}{(1-b^2)^2}, \tag{8.1}$$

$$N_b = \sum_{\substack{i,j\ge 0\\ i+j \text{ odd}}} b^{i+j} = \frac{2b}{(1-b^2)^2}. \tag{8.2}$$

The even sum occurs in R^2. Inserting $\chi(\mathfrak{p})$ changes even parity to odd parity. A ramified prime contributes 1 to the first sum and 0 to the second. Thus, with

$$D_X = \prod_{\substack{\mathrm{N}\mathfrak{p}\le Y\\ \mathfrak{p}\notin S}} \bigl(1-h_K(\mathfrak{p})+h_K(\mathfrak{p})D_b\bigr), \qquad \theta(b)=\log D_b,$$

Proposition 3.1 gives

$$S_2 = \kappa_\Omega I_W X D_X + O\bigl(X^{1/2+\varepsilon-2a\log(1-b)+o(1)}\bigr), \qquad D_X = X^{a\theta(b)+o(1)}. \tag{8.3}$$

Here and below $\varepsilon>0$ is fixed and sufficiently small.

We justify summing the error in this application. For every ideal $\mathfrak{m}$, the factor $G_\varepsilon((\mathfrak{m})_1)$ is at most $\prod_{\mathfrak{p}|\mathfrak{m}}(1+(\mathrm{N}\mathfrak{p})^{-1/2-\varepsilon/2})$. Its sum against $r_{\mathfrak{m}}$ is bounded by

$$\prod_{\substack{\mathrm{N}\mathfrak{p}\le Y\\ \mathfrak{p}\notin S}} \left(1+\frac{b}{1-b}(1+(\mathrm{N}\mathfrak{p})^{-1/2-\varepsilon/2})\right) = X^{-a\log(1-b)+o(1)}.$$

The squarefree part of a product of two resonator ideals has logarithmic norm $O_K(Y)$, so its F_ε factor is $X^{o(1)}$. The same support majorant bounds the G_ε factor of their product. This proves absolute summability and the error in (8.3). Finally, $1-h_K(\mathfrak{p})+h_K(\mathfrak{p})D_b = D_b(1+O_b((\mathrm{N}\mathfrak{p})^{-1}))$. The prime ideal theorem and Mertens' estimate therefore give the stated size of D_X.

After insertion of a prime in the resonator support, the local quotient is

$$\frac{h_K(\mathfrak{p})N_b}{1-h_K(\mathfrak{p})+h_K(\mathfrak{p})D_b} = \frac{2b}{1+b^2} + O_b((\mathrm{N}\mathfrak{p})^{-1}). \tag{8.4}$$

This exact parity count is the sharpened local factor mentioned after the corresponding calculation in [10]. It is slightly stronger than replacing the quotient by $b+o(1)$ unless b is close to one. A prime outside the support gives no square term. Primes in S have zero character value. Reinserting them in the prime sum below costs $O(1)$. Consequently,

$$\begin{aligned}&\sum_{\mathfrak{a}\in\Omega} W(\mathrm{N}\mathfrak{a}/X)P_\sigma(\chi_{\mathfrak{a}})R_{\mathfrak{a}}^2\\ &= \kappa_\Omega I_W X D_X\left(\frac{2b}{1+b^2}\sum_{\mathrm{N}\mathfrak{p}\le Y}(\mathrm{N}\mathfrak{p})^{-\sigma} + O_{K,\sigma,b}(1)\right) + O\bigl(X^{1/2+\varepsilon-2a\log(1-b)+o(1)}\bigr).\end{aligned} \tag{8.5}$$

The local error is summable because $\sum_{\mathfrak{p}}(\mathrm{N}\mathfrak{p})^{-1-\sigma}<\infty$. For the average error, the inserted prime adds at most $\log y$ to the logarithmic norm of the squarefree part. The additional sum of $(\mathrm{N}\mathfrak{p})^{-\sigma}(1+(\mathrm{N}\mathfrak{p})^{-1/2-\varepsilon/2})$ over $\mathrm{N}\mathfrak{p}\le y$ is $X^{o(1)}$. The preceding majorant therefore applies unchanged.

Since

$$\theta(b)+2\log(1-b) = \log\frac{1+b^2}{(1+b)^2} = -\frac{\alpha(b)}{2},$$

both errors are negligible whenever

$$a < \frac{1}{\alpha(b)}, \qquad 0 < \varepsilon < \frac{1 - a\alpha(b)}{2}. \tag{8.6}$$

Dividing by the common main term in (8.3) gives

$$\frac{\sum W(\mathrm{N}\mathfrak{a}/X)P_\sigma(\chi_\mathfrak{a})R_\mathfrak{a}^2}{S_2} = \left(\frac{2b\,a^{1-\sigma}}{(1+b^2)(1-\sigma)} + o(1)\right)\frac{(\log X)^{1-\sigma}}{(\log_2 X)^\sigma}.$$

We used partial summation in the prime ideal theorem to obtain $\sum_{\mathrm{N}\mathfrak{p}\le Y}(\mathrm{N}\mathfrak{p})^{-\sigma} \sim Y^{1-\sigma}/((1-\sigma)\log Y)$. For every fixed $a < 1/\alpha(b)$ we may first let $X \to \infty$ with ε chosen as in (8.6). The resulting lower bound is valid for every such fixed a. Letting $a \uparrow 1/\alpha(b)$ afterwards (or using a diagonal choice of $a = a(X)$) proves the maximum assertion.

For the frequency assertion, put $a_0 = 1/\alpha(b) - \eta > 0$ and write

$$H_\sigma(X) = (\log X)^{1-\sigma}(\log_2 X)^{-\sigma}.$$

Choose $a = a_0 + \rho_X$, where $\rho_X \to 0^+$ so slowly that $\rho_X H_\sigma(X) \to \infty$, and also $\rho_X < \frac{1}{2}(1/\alpha(b) - a_0) = \eta/2$ for large X. Thus a remains a fixed positive distance below $1/\alpha(b)$, and the error estimates above are uniform for a in a compact interval around a_0. Since the main coefficient is a positive constant times $a^{1-\sigma}$, the mean at $a_0 + \rho_X$ exceeds its value at a_0 by $\gg_{\sigma,b,\eta} \rho_X H_\sigma(X)$. Hence, after slowing ρ_X if necessary, the weighted mean exceeds $\tau_{\sigma,\eta}(X) + C_\sigma + 1$, where C_σ bounds the error in Lemma 4.3. Let $\mathcal{E} \subseteq \Omega(X)$ be the set where $\log L(\sigma, \chi_\mathfrak{a}) > \tau_{\sigma,\eta}(X)$. Outside $\mathcal{E}$ we have $P_\sigma \le \tau_{\sigma,\eta}(X) + C_\sigma$. Subtracting this contribution from the weighted mean gives

$$\sum_{\mathfrak{a}\in\mathcal{E}} W(\mathrm{N}\mathfrak{a}/X)R_\mathfrak{a}^2 \ge S_2 X^{-o(1)},$$

because $|P_\sigma| + |\tau_{\sigma,\eta}(X)| + C_\sigma = X^{o(1)}$. On the other hand,

$$R_\mathfrak{a}^2 \le (1-b)^{-2\pi_K(Y)} = X^{-2a\log(1-b)+o(1)}.$$

Combining these bounds with (8.3) and $\#\Omega(X) = X^{1+o(1)}$, we obtain

$$\frac{\#\mathcal{E}}{\#\Omega(X)} \ge X^{a\theta(b)+2a\log(1-b)+o(1)} = X^{-\frac{1}{2}(1-\eta\alpha(b))+o(1)}.$$

$\square$

9. Large values at 1

At $s = 1$, the small primes determine both the leading factor and the constant term. We use the coefficients from [10, Theorems 2 and 3]. They approach 1 at each fixed prime, which makes the resonating Euler product close to Mertens' product. The distribution of quadratic values at 1 is studied in [23, 30].

The following estimate justifies the infinite series used in the proof. It also controls products whose individual squarefree parts overlap.

Lemma 9.1. *Fix $B, c > 0$, and put $y = (\log X)^B$ and $z = c^{-1}\log X \log_2 X$. Let $a_\mathfrak{l}$ be the indicator that all prime factors of $\mathfrak{l}$ have norm at most y. Define a completely multiplicative function r by*

$$r_\mathfrak{p} = 1 - \mathrm{N}\mathfrak{p}/z \quad (\mathrm{N}\mathfrak{p} < z,\ \mathfrak{p} \notin S), \qquad r_\mathfrak{p} = 0 \quad \textit{otherwise.}$$

For every fixed $0 < \varepsilon < 1/2$, the following estimates hold:

(i)

$$\sum_\mathfrak{l} \frac{a_\mathfrak{l}}{\mathrm{N}\mathfrak{l}} F_\varepsilon((\mathfrak{l})_0)G_\varepsilon((\mathfrak{l})_1) = X^{o(1)}. \tag{9.1}$$

(ii)
$$\sum_{\mathfrak{m}} r_{\mathfrak{m}} F_\varepsilon((\mathfrak{m})_0) G_\varepsilon((\mathfrak{m})_1) = X^{1/c+o(1)}. \tag{9.2}$$

(iii) *Both estimates remain valid after replacing* $G_\varepsilon((\mathfrak{n})_1)$ *by* $\prod_{\mathfrak{p}|\mathfrak{n}}(1+(\mathrm{N}\mathfrak{p})^{-1/2-\varepsilon/2})$. *In particular,*
$$\sum_{\mathfrak{l},\mathfrak{m},\mathfrak{n}} \frac{a_{\mathfrak{l}} r_{\mathfrak{m}} r_{\mathfrak{n}}}{\mathrm{N}\mathfrak{l}} F_\varepsilon((\mathfrak{l}\mathfrak{m}\mathfrak{n})_0) G_\varepsilon((\mathfrak{l}\mathfrak{m}\mathfrak{n})_1) \le X^{2/c+o(1)}.$$

Proof. We prove the stronger bounds in the third assertion. In the first sum, the ideals of norm at most X have $F_\varepsilon = X^{o(1)}$. The remaining Euler product satisfies
$$\sum_{\mathfrak{l}} \frac{a_{\mathfrak{l}}}{\mathrm{N}\mathfrak{l}} \prod_{\mathfrak{p}|\mathfrak{l}} (1+(\mathrm{N}\mathfrak{p})^{-1/2-\varepsilon/2}) \ll_{K,\varepsilon} \prod_{\mathrm{N}\mathfrak{p}\le y} (1-(\mathrm{N}\mathfrak{p})^{-1})^{-1} \ll_{K,B,\varepsilon} \log y.$$
For $\mathrm{N}\mathfrak{l} > X$, put $\delta_X = (\log X)^{-\varepsilon/3}$. The definition of F_ε and the maximal-order bound used in Lemma 3.2 imply, uniformly in this range,
$$F_\varepsilon((\mathfrak{l})_0) \prod_{\mathfrak{p}|\mathfrak{l}} (1+(\mathrm{N}\mathfrak{p})^{-1/2-\varepsilon/2}) \le (\mathrm{N}\mathfrak{l})^{\delta_X}.$$
Indeed, their logarithms are bounded respectively by
$$O_{K,\varepsilon}((\log \mathrm{N}\mathfrak{l})^{1-\varepsilon}) \quad \text{and} \quad O_{K,\varepsilon}((\log \mathrm{N}\mathfrak{l})^{1/2-\varepsilon/2}).$$
The tail is therefore at most
$$\prod_{\mathrm{N}\mathfrak{p}\le y} (1-(\mathrm{N}\mathfrak{p})^{-1+\delta_X})^{-1} = X^{o(1)}.$$
To see the last estimate, note that $\delta_X \log y = o(1)$. The logarithm of this product is $O_K(\log\log y + 1)$ by Mertens' estimate.

Every prime factor of a resonator ideal has norm below z. Thus its squarefree part has logarithmic norm $O_K(z)$ and $F_\varepsilon = X^{o(1)}$ uniformly. The other factor sums to
$$\prod_{\substack{\mathrm{N}\mathfrak{p}<z \\ \mathfrak{p}\notin S}} \left(1+\frac{r_{\mathfrak{p}}}{1-r_{\mathfrak{p}}}(1+(\mathrm{N}\mathfrak{p})^{-1/2-\varepsilon/2})\right)$$
$$= \exp\left(\sum_{\substack{\mathrm{N}\mathfrak{p}<z \\ \mathfrak{p}\notin S}} \log\frac{z}{\mathrm{N}\mathfrak{p}} + o(\log X)\right) = X^{1/c+o(1)}.$$
Here $\sum_{\mathrm{N}\mathfrak{p}<z}(\mathrm{N}\mathfrak{p})^{-1/2-\varepsilon/2} = o(\log X)$, and
$$\sum_{\mathrm{N}\mathfrak{p}<z} \log\frac{z}{\mathrm{N}\mathfrak{p}} = \int_2^z \frac{\pi_K(t)}{t}\,dt = \frac{z}{\log z}(1+o(1)) = \frac{\log X}{c}(1+o(1)).$$
The sum without either error factor has the same exponent, which proves the matching lower bound in (9.2).

For the product assertion, its squarefree support is contained in the union of the individual squarefree supports. If a, b, c are the norms of the three individual squarefree parts, then the norm of the product's squarefree part is at most abc. The inequality $2+abc \le (2+a)(2+b)(2+c)$, followed by $(u+v)^{1-\varepsilon} \le u^{1-\varepsilon}+v^{1-\varepsilon}$, therefore bounds its F_ε factor by the product of the three individual factors, with the same defining constant $C_{K,\varepsilon}$. Every prime dividing its square part divides at least one whole ideal. The support products in the third assertion bound its G_ε factor. Multiplication of the preceding sums proves the claim. □

Proof of Theorem 1.5. Fix $c > 0$, to be specified below, and use $z = c^{-1} \log X \log_2 X$. Take the function r from Lemma 9.1, and put

$$R_{\mathfrak{a}} = \prod_{\substack{\mathrm{N}\mathfrak{p}<z \\ \mathfrak{p}\notin S}} (1 - r_{\mathfrak{p}}\chi_{\mathfrak{a}}(\mathfrak{p}))^{-1}.$$

Fix a nonzero smooth W, with $0 \le W \le 1$, supported in $(1, 2)$. Write $I_W = \int W(t)\, dt$ and

$$S_0 = \sum_{\mathfrak{a}\in\Omega} W(\mathrm{N}\mathfrak{a}/X) R_{\mathfrak{a}}^2, \qquad S_1 = \sum_{\mathfrak{a}\in\Omega} W(\mathrm{N}\mathfrak{a}/X) L(1, \chi_{\mathfrak{a}}) R_{\mathfrak{a}}^2.$$

We seek a lower bound for S_1/S_0 with an error tending to zero.

Choose $B > 1$ large enough for Lemma 4.3, so that $y = (\log X)^B > z$ for all sufficiently large X, and let

$$L_y(1, \chi) = \prod_{\mathrm{N}\mathfrak{p}\le y} (1 - \chi(\mathfrak{p})/\mathrm{N}\mathfrak{p})^{-1}.$$

Let $S_{1,y}$ denote the sum defining S_1 with L_y in place of L. Mertens' estimate gives $L_y(1, \chi) \ll_{K,B} \log_2 X$ uniformly. Lemma 4.3 consequently gives

$$\frac{S_1}{S_0} = \frac{S_{1,y}}{S_0} + o(1), \qquad L(1, \chi_{\mathfrak{a}}) \ll_{K,B} \log_2 X.$$

The latter bound will also suffice for the frequency assertion. The primes in S have zero ideal coefficient and contribute no Euler factor.

Expand the Euler products, separating the primes in S. Proposition 3.1 and Lemma 9.1 give

$$S_0 = \kappa_\Omega I_W X \sum_{\mathfrak{m}\mathfrak{n}=\square} r_{\mathfrak{m}} r_{\mathfrak{n}} h_K(\mathfrak{m}\mathfrak{n}) + O(X^{1/2+\varepsilon+2/c+o(1)}), \tag{9.3}$$

$$S_{1,y} = \kappa_\Omega I_W X \sum_{\mathfrak{l}\mathfrak{m}\mathfrak{n}=\square} \frac{a_{\mathfrak{l}} r_{\mathfrak{m}} r_{\mathfrak{n}}}{\mathrm{N}\mathfrak{l}} h_K(\mathfrak{l}\mathfrak{m}\mathfrak{n}) + O(X^{1/2+\varepsilon+2/c+o(1)}). \tag{9.4}$$

All ideals in these sums are prime to S. The main terms are nonnegative. In the second sum, retain only ideals $\mathfrak{l}$ whose prime factors have norm below z.

For $q = \mathrm{N}\mathfrak{p} < z$, $\mathfrak{p} \notin S$, and $r = 1 - q/z$, the resulting local factors are

$$D_{\mathfrak{p}} = 1 - h_K(\mathfrak{p}) + h_K(\mathfrak{p}) \frac{1 + r^2}{(1 - r^2)^2},$$
$$N_{\mathfrak{p}} = 1 - h_K(\mathfrak{p}) + \frac{h_K(\mathfrak{p})}{2} \left(\frac{1}{(1 - q^{-1})(1 - r)^2} + \frac{1}{(1 + q^{-1})(1 + r)^2} \right). \tag{9.5}$$

These formulas average the two unramified signs and the ramified value 0. They count all square terms in the two expansions.

We first compare the second moment with the averaging error. Since $h_K(\mathfrak{p}) = q/(q + 1)$,

$$D_{\mathfrak{p}} = \frac{1 + r^2}{(1 - r^2)^2} \left(1 - \frac{3r^2 - r^4}{(q + 1)(1 + r^2)} \right).$$

The logarithm of the product of the second factors is $O_K(\log \log z)$. Partial summation in the prime ideal theorem gives

$$\prod_{\substack{\mathrm{N}\mathfrak{p}<z \\ \mathfrak{p}\notin S}} \frac{1 + r_{\mathfrak{p}}^2}{(1 - r_{\mathfrak{p}}^2)^2} = X^{\lambda/c+o(1)}, \qquad \lambda = \int_0^1 \log \frac{2 - 2u + u^2}{u^2(2 - u)^2}\, du = 2 - 3 \log 2 + \frac{\pi}{2}. \tag{9.6}$$

The logarithmic singularity at $u = 0$ is integrable. To justify partial summation there, split at $\mathrm{N}\mathfrak{p} = \delta z$. The lower range contributes $O_K(\delta(1 + |\log \delta|)z/ \log z + \log z)$. The upper range uses the prime ideal theorem uniformly. First let $z \to \infty$, then $\delta \to 0^+$. The integral follows from $\int_0^1 \log(1 + t^2)\, dt = \log 2 - 2 + \pi/2$ and elementary integration of $\log u$ and $\log(2 - u)$.

Thus the error in (9.3) is smaller by a fixed power of X if

$$c > c_0 := 2(3\log 2 - \pi/2), \qquad 0 < \varepsilon < \frac{1}{2} - \frac{2-\lambda}{c}. \tag{9.7}$$

In this range,

$$S_0 = \kappa_\Omega I_W X \prod_{\substack{\mathrm{N}\mathfrak{p}<z\\ \mathfrak{p}\notin S}} D_\mathfrak{p}(1+o(1)) = X^{1+\lambda/c+o(1)}. \tag{9.8}$$

The relative error tends to zero as a fixed negative power of X. The same error bound applies to (9.4).

It remains to compute the quotient accurately enough to retain its constant term. Cancelling the common factor $h_K(\mathfrak{p})(1+r^2)/(1-r^2)^2$ gives the exact identity

$$(1-q^{-1})\frac{N_\mathfrak{p}}{D_\mathfrak{p}} = 1 - \frac{\dfrac{(1-r)^2}{(q+1)(1+r^2)} + \dfrac{(1-r^2)^2}{q^2(1+r^2)}}{1+\dfrac{(1-r^2)^2}{q(1+r^2)}}. \tag{9.9}$$

Putting $u = q/z$, we obtain, uniformly for $2 \le q < z$,

$$(1-q^{-1})\frac{N_\mathfrak{p}}{D_\mathfrak{p}} = 1 - \frac{1}{z}\frac{u}{2-2u+u^2} + O(z^{-2}).$$

Indeed, replacing $q+1$ by q costs $O(u^2/q^2) = O(z^{-2})$, and the second term in the numerator is also $O(z^{-2})$. The denominator perturbation itself is $O(u/z)$, while the leading numerator perturbation is $O(u/z)$. Hence its effect after division is $O(u^2/z^2) = O(z^{-2})$. Summing logarithms and using the prime ideal theorem now gives

$$\log \prod_{\substack{\mathrm{N}\mathfrak{p}<z\\ \mathfrak{p}\notin S}} (1-(\mathrm{N}\mathfrak{p})^{-1})\frac{N_\mathfrak{p}}{D_\mathfrak{p}} = -\frac{c_3+o(1)}{\log z}, \qquad c_3 = \int_0^1 \frac{u}{2-2u+u^2}\,du = \frac{\pi}{4} - \frac{\log 2}{2}.$$

For this use of partial summation the integrand is continuous on $[0,1]$. The omitted fixed primes and the summed $O(z^{-2})$ errors contribute $o(1/\log z)$.

For fixed K, Mertens' product and the classical zero-free region for ζ_K give the stronger remainder

$$\prod_{\substack{\mathrm{N}\mathfrak{p}<z\\ \mathfrak{p}\notin S}} (1-(\mathrm{N}\mathfrak{p})^{-1})^{-1} = e^\gamma \kappa_K \log z \prod_{\mathfrak{p}\in S}(1-(\mathrm{N}\mathfrak{p})^{-1}) \left(1+O_K(e^{-c_K\sqrt{\log z}})\right)$$

for some $c_K > 0$ (a possible exceptional real zero is harmless because K is fixed). For instance, this follows by partial summation from the fixed-field prime ideal theorem $\psi_K(x) = x + O_K(xe^{-c_K\sqrt{\log x}})$, after adjusting c_K. In particular the relative error is $o(1/\log z)$, which is the precision needed to retain the additive constant below. See [21] for number field Mertens formulas. Combining the last two displays, and (9.3)–(9.4) yields

$$\frac{S_1}{S_0} \ge \mathcal{M}_{K,\Omega}\big(\log_2 X + \log_3 X - c_3 - \log c + o(1)\big). \tag{9.10}$$

Here $\log z = \log_2 X + \log_3 X - \log c$. To preserve the fixed power saving required in (9.7), first fix any $c > c_0$ and let $X \to \infty$. The resulting lower bound holds for every such fixed c. We may then let $c \downarrow c_0$ (equivalently, use a diagonal sequence if one wants a single choice depending on X). Since $C_2 = c_3 + \log c_0$, this proves the maximum assertion.

For the frequency assertion, first fix $0 < \eta' < \eta$ and put $c = c_0 e^{\eta'}$. The lower bound in (9.10) exceeds the required threshold by $\mathcal{M}_{K,\Omega}(\eta - \eta' + o(1))$. Let $\mathcal{H}_\eta \subseteq \Omega(X)$ be the set where that threshold is exceeded. Subtracting the complementary contribution to S_1, and using $L(1,\chi) \ll \log_2 X$, gives

$$\sum_{\mathfrak{a}\in\mathcal{H}_\eta} W(\mathrm{N}\mathfrak{a}/X)R_{\mathfrak{a}}^2 \geq S_0 X^{-o(1)}.$$

The pointwise bound

$$R_{\mathfrak{a}}^2 \leq \prod_{\substack{\mathrm{N}\mathfrak{p}<z \\ \mathfrak{p}\notin S}} (z/\mathrm{N}\mathfrak{p})^2 = X^{2/c+o(1)}$$

and (9.8) consequently imply

$$\frac{\#\mathcal{H}_\eta}{\#\Omega(X)} \geq X^{-(2-\lambda)/c+o(1)} = X^{-e^{-\eta'}/2+o(1)}.$$

For each fixed $0 < \eta' < \eta$ this bound holds with a fixed power saving in (9.7). Letting $\eta' \uparrow \eta$ after $X \to \infty$, and diagonalizing over a sequence $\eta'_j \uparrow \eta$ if a single X-dependent choice is desired, gives $X^{-e^{-\eta}/2+o(1)}$. The gap in (9.7) remains bounded away from zero because the limiting value η is fixed and positive. □

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