MORDELL–LANG PLUS GEOMETRIC BOGOMOLOV IN ARBITRARY CHARACTERISTICS

LONG LIU

ABSTRACT. We prove a geometric version of Poonen's "Mordell–Lang plus Bogomolov" theorem for semi-abelian varieties in arbitrary characteristics. This is a generalization of the Mordell–Lang conjecture and the geometric Bogomolov conjecture.

CONTENTS

1. Introduction 1
2. Heights and bounded approximation 4
3. A consequence of finitely generated Mordell–Lang 12
4. Bounded replacement 17
5. Descent of constant images 24
6. Proof of the main theorem 26
References 27

1. INTRODUCTION

The combined Mordell–Lang and Bogomolov problem asks which subvarieties contain a Zariski dense set of points arbitrarily close, in canonical height, to the division points of a finitely generated subgroup. Over a function field, the conclusion must allow subvarieties coming from the constant field. We consider this problem for semiabelian varieties over function fields of arbitrary characteristic.

For arithmetic heights, Poonen proved the combined assertion for almost split semiabelian varieties over number fields, namely those isogenous to a product of an abelian variety and a torus [Poo99, Corollary 9]. Rémond treated arbitrary semiabelian varieties over $\overline{\mathbb{Q}}$ and finite-rank subgroups [Rém03, Theorem 1.1]. Moriwaki proved an abelian version over finitely generated extensions of $\mathbb{Q}$ equipped with big arithmetic polarizations [Mor01, Theorem A]. For geometric heights, the rank-zero case is the geometric Bogomolov theorem of Xie–Yuan for abelian varieties [XY22, Theorem 1.1] and of Luo–Yu for semiabelian varieties [LY25, Theorem 1.1], both in arbitrary characteristic. Liu proved the combined geometric assertion in characteristic zero [Liu26, Theorem 1.1].

The additional construction here replaces arbitrary approximating points by bounded-height approximating points after one independent extension of the constant field. Both the

Date: November 19, 2026.
2020 *Mathematics Subject Classification.* Primary 14G40; Secondary 11G50, 11J25, 14K12.
Key words and phrases. Mordell–Lang conjecture, geometric Bogomolov conjecture, semi-abelian varieties, canonical heights, function fields.

extension and the height bound are independent of the approximation threshold, and the division group is unchanged. This is Proposition 4.2. It supplies the step for which a bound on the original approximating points is unavailable in positive characteristic.

1.1. **Semiabelian varieties and heights.** Let k be an algebraically closed field, and let K/k be a finitely generated extension of transcendence degree $d \geq 1$. Fix an algebraic closure $\overline{K}$ of K. A polarization of K/k is a pair (S, M), where S is a normal projective k-variety with function field K, and M is an ample line bundle on S. All heights are taken with respect to this polarization and are normalized by degrees of finite extensions.

Let G be a semiabelian variety over K. Thus there is an exact sequence

$$0 \longrightarrow T \longrightarrow G \xrightarrow{\pi} A \longrightarrow 0,$$

where T is a torus and A is an abelian variety. After a finite extension of K, we may assume that $T \simeq \mathbb{G}_m^{r_T}$. Here r_T is the rank of the torus. The usual action of T on $(\mathbb{P}^1)^{r_T}$ gives the associated compactification $\overline{G} \to A$; its fibers are products of projective lines. Let D be the sum of the zero and infinity boundary divisors, and let L be a symmetric ample line bundle on A.

Write $\widehat{h}_D$ for the canonical height of the boundary divisor and $\widehat{h}_L$ for the Néron–Tate height on A. More explicitly, if h_D and h_L are Weil heights for these bundles, then

$$\widehat{h}_D(x) = \lim_{n\to\infty} \frac{h_D([n]x)}{n}, \qquad \widehat{h}_L(a) = \lim_{n\to\infty} \frac{h_L([n]a)}{n^2}.$$

Here $[n]$ denotes multiplication by the integer n, $x \in G(\overline{K})$, and $a \in A(\overline{K})$. The model construction and the independence of these limits from the chosen Weil heights are recalled in [LY25, Section 2].

We use the nonnegative height

$$\widehat{h}(x) = \widehat{h}_D(x) + \widehat{h}_L(\pi(x)).$$

The two terms have different homogeneity degrees. For approximation it is therefore convenient to use

$$\rho(x) = \widehat{h}_D(x) + \sqrt{\widehat{h}_L(\pi(x))}.$$

This function is homogeneous of degree one and satisfies the triangle inequality. Moreover,

$$\rho(x) \leq \widehat{h}(x) + \sqrt{\widehat{h}(x)}, \qquad \widehat{h}(x) \leq \rho(x) + \rho(x)^2. \tag{1}$$

Thus either function can be used to express boundedness or convergence to zero. The relevant height comparisons are given in §2.

1.2. **The combined assertion.** Let $\Gamma_0 \subset G(\overline{K})$ be a finitely generated subgroup. Its full division hull is

$$\Gamma = \{x \in G(\overline{K}) : [n]x \in \Gamma_0 \text{ for some integer } n \geq 1\}.$$

The integer n may be divisible by the characteristic. In particular, Γ contains every torsion point. For $\epsilon > 0$, put

$$\Gamma_\epsilon = \{\gamma + z : \gamma \in \Gamma,\ z \in G(\overline{K}),\ \widehat{h}(z) < \epsilon\}.$$

This is the set of points whose error from a division point has height less than ϵ.

For a semiabelian variety H over $\overline{K}$, write $H^{\overline{K}/k}$ for its Chow $\overline{K}/k$-trace, a semiabelian variety over k, and let

$$\mathrm{tr}_H : (H^{\overline{K}/k})_{\overline{K}} \longrightarrow H$$

be the canonical homomorphism. Its universal property says that every homomorphism $u : C_{\overline{K}} \to H$, with C semiabelian over k, factors uniquely as $u = \mathrm{tr}_H \circ v_{\overline{K}}$ for a homomorphism $v : C \to H^{\overline{K}/k}$ over k. Existence follows from [Liu24, Corollary 2.4.4(2)] and [Liu]. The trace map has finite kernel, possibly nonreduced; the argument is recalled in §2.2.

Let $X \subset G_{\overline{K}}$ be a closed integral subvariety. Write $B = \mathrm{Stab}_G(X)^0_{\mathrm{red}}$ for the connected reduced stabilizer, set $Q = G_{\overline{K}}/B$, and let $q : G_{\overline{K}} \to Q$ be the quotient map. We say that X is Γ-*special* if

$$q(X) = q(\gamma) + \mathrm{tr}_Q(Y_{\overline{K}}) \tag{2}$$

for some $\gamma \in \Gamma$ and some closed integral k-subvariety $Y \subset Q^{\overline{K}/k}$. The quotient removes the directions along which translation leaves X unchanged. Its Chow trace accounts for homomorphisms from constant semiabelian varieties, so the remaining subvariety comes from k, up to translation by $q(\gamma)$.

The combined assertion is the following.

Theorem 1.1. *With the notation above, the following conditions are equivalent.*

(1) For every $\epsilon > 0$, the set $X(\overline{K}) \cap \Gamma_\epsilon$ is Zariski dense in X.
(2) The subvariety X is Γ-special.

For use in the proof, define

$$d_\Gamma(x) = \inf_{\gamma \in \Gamma} \rho(x - \gamma).$$

By (1), condition (1) is equivalent to density of $\{x \in X(\overline{K}) : d_\Gamma(x) < \epsilon\}$ for every $\epsilon > 0$. The function d_Γ will allow us to translate approximation estimates into norm estimates on a finite-dimensional real vector space.

When $\Gamma_0 = 0$, the assertion is geometric Bogomolov: translation by torsion does not change the height. Exact intersection points in $X \cap \Gamma$ satisfy condition (1) with zero error. The arbitrary-characteristic assertion therefore includes a finite-rank Mordell–Lang conclusion. Only the theorem for finitely generated subgroups is used as an input below.

For semiabelian varieties, dense small points must not be replaced by dense points of height zero. In a nonsplit extension of a constant elliptic curve by $\mathbb{G}_m$, a special fiber can contain points of arbitrarily small height and no points of height zero; see [LY25, Example 1.2]. The reverse implication of Theorem 1.1 therefore uses geometric Bogomolov; it does not require lifts of height zero.

1.3. **The argument.** Suppose first that the approximating points have bounded height. Their chosen centers in Γ, modulo the kernel of ρ, lie in a bounded subset of a finite-dimensional real normed space. Covering this subset by small balls produces dense sets whose pairwise differences have arbitrarily small height. Following the difference-variety method of Moriwaki [Mor01, Theorem 2.5 and Section 3.2], geometric Bogomolov identifies the subvariety as a translate of a constant image. A height gap over each fixed finite extension of K then places the translating point in Γ, up to a constant point. This proves the bounded case in §2.

In positive characteristic, the original points need not have bounded height. Over $k(t)$ in characteristic p, the curve $u+v=1$ in $\mathbb{G}_m^2$ contains all the points $[p^n](t,1-t)$. They belong to a cyclic subgroup and have unbounded height. Thus the height bound used in the characteristic-zero argument cannot hold in this form; see [Liu26, Section 5.3]. Bounded replacement preserves the required density while allowing different points on the same subvariety.

To construct these replacements, subtract the integer parts of the rational coefficients expressing each approximation center in terms of fixed generators of Γ_0. The remainders have bounded height, so their linear spaces of bounded-degree equations belong to a family over k. Independent evaluations over an extension of k detect all the relevant translation conditions simultaneously. The integer parts then lie in a finitely generated subgroup of a product whose first factor is constant. Proposition 3.3 describes the resulting dense intersection as a division-point translate of a constant image; constant points in its fibers give the bounded replacements. In positive characteristic, the required control of the translating point follows by applying Laurent's theorem to Frobenius exponent tuples. A nonnegative ray in the resulting lattice coset, together with an integer recurrence, places that translating point in the full division hull.

The product statement is proved in §3, and the replacement construction in §4. Descent of constant images in §5 returns the conclusion to the original field. Section 6 combines bounded replacement, the bounded case, and descent to prove Theorem 1.1 and its finite-union consequence.

Conventions. A subvariety is given its reduced structure; an integral subvariety is irreducible and reduced. Kernels of homomorphisms are taken as group schemes. Thus a finite kernel need not be reduced. For a variety V over a field E, write $V_F = V \otimes_E F$ after an extension F/E, and omit the subscript when the base is clear. We may replace K by a finite extension so that G, X, the chosen line bundles, and the generators of Γ_0 are defined over K. Pulling back the polarization and retaining the original degree normalization leaves the heights unchanged; renormalizing over the new field changes all heights by one fixed positive factor. Neither choice changes the assertion.

2. Heights and bounded approximation

The first part of the argument concerns approximating points of bounded height. We begin with the height comparisons needed to pass between a semiabelian variety, its subvarieties, and its quotients. We then use difference maps to reduce bounded approximation to geometric Bogomolov.

2.1. **Height comparisons.** The canonical boundary height satisfies the triangle inequality. One obtains this from the effective-divisor comparison on the compactified addition graph in [Rém03, Corollary 3.1]: apply the associated Weil-height inequality to integer multiples and divide by the integer before taking the limit. More precisely, if p_1, p_2 and s are the two projections and addition map on that graph, the divisor $p_1^*D + p_2^*D - s^*D$ has a section nonvanishing on the open graph. Its height inequality has a uniform additive constant. Evaluation at $([n]x,[n]y)$, followed by division by n, removes this constant in the canonical limit. This construction is algebraic and applies in every characteristic. Together with the

Néron–Tate triangle inequality, it gives

$$\rho(x+y) \le \rho(x) + \rho(y), \qquad \rho([n]x) = |n|\rho(x) \quad (n \in \mathbb{Z}).$$

In particular, ρ vanishes on torsion points and is unchanged by translation by such points.

The bundle $\mathcal{O}(D) \otimes \overline{\pi}^* L$ is ample on $\overline{G}$, where $\overline{\pi} : \overline{G} \to A$ extends π. Indeed, the extension line bundles on A are numerically trivial, the boundary bundle is nef and relatively ample, and L is ample on the base. The identities $[n]^* D = nD$ and $[n]^* L = n^2 L$, for $n \ge 1$, show by Tate's limiting argument that $\widehat{h}$ differs by a bounded function from a Weil height for this ample bundle. Thus a set has bounded $\widehat{h}$ if and only if it has bounded projective height in any fixed projective embedding of $\overline{G}$.

The next comparison is used for the trace map. Its second assertion includes maps whose images are proper subgroups of the target, so we give the reduction to the standard upper comparison.

Lemma 2.1. *Let $f : G_1 \to G_2$ be a homomorphism of semiabelian varieties over $\overline{K}$. For $i = 1, 2$, let D_i be the standard boundary divisor, let L_i be a symmetric ample line bundle on the abelian quotient $\pi_i : G_i \to A_i$, and put*

$$\widehat{h}_i = \widehat{h}_{D_i} + \widehat{h}_{L_i} \circ \pi_i.$$

Then the following assertions hold.

(1) There is a real number $c > 0$ such that $\widehat{h}_2(f(x)) \le c\widehat{h}_1(x)$ for all $x \in G_1(\overline{K})$.

(2) If $\ker f$ is finite, there is a real number $c' > 0$ such that $\widehat{h}_1(x) \le c'\widehat{h}_2(f(x))$ for all $x \in G_1(\overline{K})$.

Proof. Write $f_{\mathrm{tor}} : T_1 \to T_2$ and $f_{\mathrm{ab}} : A_1 \to A_2$ for the induced homomorphisms. As in [LY25, Lemma 3.3], compare the boundary and abelian heights separately. In split torus coordinates, let (a_{ij}) be the matrix of f_{tor}, with i indexing the target coordinates and j the source coordinates. Choose an integer $c_T \ge 1$ at least as large as every column sum $\sum_i |a_{ij}|$.

Let p_1, p_2 be the projections from the normalization of the closure of the graph of f in $\overline{G}_1 \times \overline{G}_2$. At a prime divisor of this graph, choose local trivializations of the two torus torsors near the images of its generic point in A_1 and A_2. Write v_j for the order of the j-th source coordinate. By f_{tor}-equivariance, the i-th target coordinate pulls back to the monomial with exponents a_{ij} times a unit from the base. Its order is therefore $\sum_j a_{ij} v_j$. The sum of the zero and infinity divisors in one coordinate has multiplicity equal to the absolute value of its order. Hence

$$\sum_i \Big| \sum_j a_{ij} v_j \Big| \le \sum_j \Big(\sum_i |a_{ij}| \Big) |v_j| \le c_T \sum_j |v_j|.$$

It follows that $c_T p_1^* D_1 - p_2^* D_2$ is effective and supported outside the open graph. Its Weil height is bounded below on the open graph. Evaluating at $([n]x, f([n]x))$, dividing by n, and passing to the limit gives $\widehat{h}_{D_2}(f(x)) \le c_T \widehat{h}_{D_1}(x)$.

Choose $c_A \ge 1$ integral so that $L_1^{\otimes c_A} \otimes (f_{\mathrm{ab}}^* L_2)^{-1}$ is ample. Its nonnegative Néron–Tate height gives $\widehat{h}_{L_2}(f_{\mathrm{ab}}(a)) \le c_A \widehat{h}_{L_1}(a)$. Adding the two comparisons proves the first assertion with $c = \max\{c_T, c_A\}$.

Suppose now that $\ker f$ is finite. Then f_{tor} has finite kernel. To see that f_{ab} also has finite kernel, let $B_1 = (\ker f_{\mathrm{ab}})^0_{\mathrm{red}}$ and let G_{B_1} be its inverse image in G_1. The homomorphism $G_{B_1} \to T_2$ has finite kernel, so it is finite onto a closed subgroup of a torus. Thus G_{B_1} is affine. Its abelian quotient B_1 must be zero. Hence $\ker f_{\mathrm{ab}}$ is finite, and $f_{\mathrm{ab}}^* L_2$ is ample on A_1.

Set $E = A_1 \times_{A_2} G_2$. The homomorphism f factors as

$$G_1 \xrightarrow{j} E \xrightarrow{p} G_2,$$

where j induces f_{tor} on tori and the identity on A_1. Equip E with its standard boundary divisor and with the ample abelian bundle $f_{\mathrm{ab}}^* L_2$, and denote the resulting height by $\widehat{h}_E$. Its standard compactification is the pullback of the standard compactification of G_2. Functoriality of the canonical heights gives

$$\widehat{h}_E(j(x)) = \widehat{h}_2(f(x)).$$

The matrix of f_{tor} has full column rank. A rational left inverse, after clearing denominators, gives a homomorphism $r_{\mathrm{tor}} : T_2 \to T_1$ and an integer $n \geq 1$ such that

$$r_{\mathrm{tor}} \circ f_{\mathrm{tor}} = [n]_{T_1}.$$

Let $\xi \in \mathrm{Ext}^1(A_1, T_1)$ be the extension class of G_1. The extension class of E is $(f_{\mathrm{tor}})_* \xi$. Consequently

$$(r_{\mathrm{tor}})_* (f_{\mathrm{tor}})_* \xi = n\xi = [n]_{A_1}^* \xi.$$

The classification of extensions [LY25, §3.1] therefore supplies a homomorphism $r : E \to G_1$ inducing r_{tor} on tori and $[n]$ on the abelian quotient. The maps rj and $[n]_{G_1}$ induce the same maps on the torus and the abelian quotient. Their difference factors through a homomorphism $A_1 \to T_1$, which is zero because A_1 is proper and T_1 is affine. Hence $rj = [n]_{G_1}$.

Apply the first assertion to r. For some constant $c_r > 0$,

$$n\widehat{h}_1(x) \leq \widehat{h}_1([n]x) = \widehat{h}_1(rj(x)) \leq c_r \widehat{h}_E(j(x)) = c_r \widehat{h}_2(f(x)).$$

The first inequality uses the respective scaling factors n and n^2 of the boundary and abelian summands. This proves the second assertion. □

2.2. **The Chow trace and the stabilizer.** We use G also for its base change to $\overline{K}$, and abbreviate $C = G^{\overline{K}/k}$ and $\mathrm{tr} = \mathrm{tr}_G$. The Chow trace map $\mathrm{tr} : C_{\overline{K}} \to G$ has finite kernel. Indeed, its connected reduced kernel descends to a semiabelian subvariety $C_0 \subset C$ by [Liu24, Theorem 2.3.12]. The inclusion and the zero map $C_0 \to C$ both factor the zero homomorphism $(C_0)_{\overline{K}} \to G$ through the trace. Uniqueness forces the inclusion to be zero. Thus the kernel is zero-dimensional and hence finite. This argument applies to every semiabelian variety over $\overline{K}$, in particular to the quotient Q in §1.2. We write H for the image of tr. In positive characteristic the finite kernel may be nonreduced, and H need not be isomorphic to a constant semiabelian variety. The trace of a product is the product of the traces.

The definition (2) is equivalent to allowing any finite-kernel homomorphism from a constant semiabelian variety to Q. Indeed, let $u : C'_{\overline{K}} \to Q$ be such a homomorphism and let $Y' \subset C'$ be a closed integral subvariety over k. The universal property gives a homomorphism $v : C' \to Q^{\overline{K}/k}$ over k with $u = \mathrm{tr}_Q \circ v_{\overline{K}}$. Since $\ker v_{\overline{K}} \subset \ker u$, the homomorphism v has finite kernel and is finite onto its image. Hence $Y = v(Y')$, with its reduced structure, is closed and integral over k, and $u(Y'_{\overline{K}}) = \mathrm{tr}_Q(Y_{\overline{K}})$. Conversely, tr_Q itself has finite kernel. The definition uses the Chow trace of Q; no compatibility of the Chow trace with quotients is required.

A homomorphism from a constant semiabelian variety sends its constant points to points of height zero. Conversely, geometric Bogomolov for a point, together with the universal property of the trace, gives

$$\ker \rho = G(\overline{K})_{\mathrm{tors}} + \mathrm{tr}(C(k)). \tag{3}$$

Here $G(\overline{K})_{\mathrm{tors}}$ is the torsion subgroup. The formula describes points of height zero in G; it does not assert that such points lift through an arbitrary quotient.

For the forward implication of Theorem 1.1, we may replace G and X by G/B and $q(X)$, where B and q are defined in the introduction. The image $q(X)$ is closed because the B-invariant closed subvariety X descends through the faithfully flat quotient. Its stabilizer is finite. Lemma 2.1 preserves small errors under q, and

$$q(\Gamma_0^{\mathrm{div}}) = (q\Gamma_0)^{\mathrm{div}}. \tag{4}$$

Indeed, if $[n]\overline{a} = q(g)$ with $g \in \Gamma_0$, lift $\overline{a}$ to $a \in G(\overline{K})$. Then $[n]a - g \in B(\overline{K})$. Divisibility of $B(\overline{K})$ gives $b \in B(\overline{K})$ with $[n]b = [n]a - g$. Hence $a - b \in \Gamma$ lifts $\overline{a}$. This also lifts the translating point when we return from the quotient to G.

2.3. **Approximation of a point.** The first observation is that positive heights of points in a fixed finite extension are bounded away from zero. On the abelian quotient this follows from Lang–Néron. Above a point coming from the trace, the toric height is the degree of an effective divisor on the base.

Lemma 2.2. *For every finite extension E/K, there is a real number $a_E > 0$ such that*

$$P \in G(E), \qquad \widehat{h}(P) < a_E \quad \Longrightarrow \quad \widehat{h}(P) = 0.$$

Proof. After replacing E by a finite extension, we may assume that the torus is split over E and that the height bundles are defined over E. A lower bound for points over the larger field is also a lower bound for the original points.

Let $(A_0, \mathrm{tr}_{E/k})$ be the E/k-trace of A. By Lang–Néron [Con06, Theorem 2.1], the group

$$A(E)/\mathrm{tr}_{E/k}(A_0(k))$$

is finitely generated. The canonical quadratic height is positive definite on its real vector space modulo torsion; see [Con06, Lemma 9.13 and Theorem 9.15]. Thus its positive values on this group are bounded below by a number $b_E > 0$. Choose an integer $e \geq 1$ annihilating the torsion of the quotient. If $\widehat{h}(P) < b_E$, then $\widehat{h}_L(\pi(P)) = 0$, and

$$[e]\pi(P) = \mathrm{tr}_{E/k}(a) \quad \text{for some } a \in A_0(k). \tag{5}$$

Pull back $G_E \to A_E$ to $(A_0)_E$. The resulting extension is described by finitely many points $\eta_j \in A_0^\vee(E)$, one for each torus coordinate. Let S_E be the normalization of S in E, and let M_E be the pullback of M. Properness of $A_0^\vee$ extends the rational maps corresponding to the η_j over an open subset $U \subset S_E$ whose complement has codimension at least two.

The rigidified Poincaré bundle on $A_0 \times A_0^\vee$, pulled back along these maps, defines the extension and its standard compactification over U. Denote its boundary divisor by D. The rigidified multiplication identities give

$$[n]^* D = nD \qquad (n \geq 1). \tag{6}$$

Here multiplication covers $[n]$ on A_0, and the projective-line coordinates are raised to their n-th powers.

Let P' be an E-rational point of the pulled-back extension above a point of $A_0(k)$. Since $E = k(S_E)$, it gives a rational section of the compactification over U. Properness extends this section at every codimension-one point of S_E. Pulling back the boundary there defines an effective integral Weil divisor $D_{P'}$ on S_E. With the original K-normalization, the model boundary height of P' is

$$\frac{\deg_{M_E}(D_{P'})}{[E:K]}. \tag{7}$$

Here $\deg_{M_E}(D_{P'}) = (c_1(M_E)^{d-1} \cdot D_{P'})$. Completing the model over the omitted set preserves this codimension-one formula, which gives a Weil height for the boundary bundle up to a bounded function. By (6), the displayed height at $[n]P'$ is exactly n times its value at P'. The boundary on the generic fiber is the pullback of the boundary of $\overline{G}_E$. Thus functoriality and the canonical limit identify (7) with the canonical boundary height of the image of P' in G.

The degree of a nonzero effective integral divisor with respect to the ample line bundle is a positive integer. Apply this observation to $P' = ([e]P, a)$, which is an E-rational point of the pullback by (5) and maps to $[e]P$. Since $\widehat{h}_D([e]P) = e\widehat{h}_D(P)$, either $\widehat{h}_D(P) = 0$ or

$$\widehat{h}_D(P) \geq \frac{1}{e[E:K]}.$$

Any positive number smaller than both b_E and $1/(e[E:K])$ has the required property. □

The height gap also identifies the points that can be approximated by Γ. The only possible error is a point of height zero, hence a trace point modulo torsion.

Lemma 2.3. *The following assertions hold.*

(1) For every finitely generated subgroup $\Lambda \subset G(\overline{K})$, the kernel of the continuous extension of ρ to $\Lambda \otimes_{\mathbb{Z}} \mathbb{R}$ is defined over $\mathbb{Q}$.

(2) If $a \in G(\overline{K})$ satisfies $d_\Gamma(a) = 0$, then

$$a \in \Gamma + \mathrm{tr}(C(k)). \tag{8}$$

Proof. Homogeneity defines ρ on $\Lambda \otimes \mathbb{Q}$. If $g_1, \dots, g_s$ generate Λ modulo torsion, the triangle inequality gives

$$\rho\Big(\sum_i t_i g_i\Big) \leq \sum_i |t_i| \rho(g_i) \qquad (t_i \in \mathbb{Q}).$$

It follows that ρ extends uniquely to a continuous seminorm on $V = \Lambda \otimes \mathbb{R}$. Write W for its kernel.

All generators are defined over a finite extension E/K. Lemma 2.2 and (1) give a positive lower bound for the nonzero values of ρ on Λ. Thus the image of Λ in V/W is discrete. Since it spans this real vector space, it is a lattice of rank $\dim(V/W)$. Consequently the kernel on $\Lambda/\Lambda_{\mathrm{tors}}$ has rank $\dim W$ and spans W. This proves the first assertion.

For the second, take $\Lambda = \langle a, g_1, \dots, g_r \rangle$, where $g_1, \dots, g_r \in \Gamma_0$ span $\Gamma \otimes \mathbb{Q}$. The hypothesis $d_\Gamma(a) = 0$ places the image of a in the closure of the span of the images of the g_i in V/W. This span is closed. Both it and W are defined over $\mathbb{Q}$, so there are $c_1, \dots, c_r \in \mathbb{Q}$ such that

$$\rho\Big(a - \sum_i c_i g_i\Big) = 0.$$

Choose an integer $n \geq 1$ clearing the denominators, and choose $\gamma \in G(\overline{K})$ with $[n]\gamma = \sum_i n c_i g_i$. Then $\gamma \in \Gamma$, and homogeneity gives $\widehat{h}(a - \gamma) = 0$. By (3), the point $a - \gamma$ is the sum of

a torsion point and an element of $\mathrm{tr}(C(k))$. Since Γ contains every torsion point, this proves (8). □

2.4. **Difference maps.** Differences remove the common center of an approximating tuple. For an integral closed subvariety $X \subset G$ and an integer $m \geq 2$, put

$$\alpha_m : X^m \longrightarrow G^{m-1}, \qquad (x_1, \ldots, x_m) \longmapsto (x_2 - x_1, \ldots, x_m - x_1), \qquad W_m = \overline{\alpha_m(X^m)}.$$

The closure is needed because G need not be proper.

Lemma 2.4. *Suppose that X has finite stabilizer and $m > \dim G$, with $m \geq 2$. Then*

(1) α_m is generically finite onto W_m;
(2) W_m has finite stabilizer.

Proof. There is nothing to prove if X is a point. Put $d = \dim X > 0$, $g = \dim G$, and $J = \mathrm{Stab}_G(X)(\overline{K})$. For $t \notin J$, the intersection $X \cap (X - t)$ has dimension at most $d-1$. The incidence of tuples $(x_1, \ldots, x_m, t)$ with $t \notin J$ and $x_i + t \in X$ for all i therefore has dimension at most

$$g + m(d-1) < md = \dim X^m.$$

Hence a general tuple admits only translations in the finite set J. Its fiber under α_m is finite.

Let $v = (v_2, \ldots, v_m)$ stabilize W_m, and put $v_1 = 0$. The image of α_m contains a dense open subset of W_m. Intersecting this open subset with its translate by $-v$, we see that a general tuple $(x_i) \in X^m$ admits a point $t \in G$ such that $x_i + t + v_i \in X$ for every i. Suppose that $v \notin J^{m-1}$. Outside the finite set $\bigcup_i (J - v_i)$, the preceding incidence has dimension at most $g + m(d-1)$. For a fixed t in that finite set, at least one $t + v_i$ is outside J: otherwise $v_1 = 0$ would imply $t \in J$ and $v \in J^{m-1}$. The incidence over such a t has dimension at most $md - 1$. Neither part can dominate X^m, a contradiction. Thus $\mathrm{Stab}(W_m)(\overline{K}) \subset J^{m-1}$. □

The next lemma recovers a translate of X from two consecutive difference images. Passing to the trace before taking a fiber ensures that the fiber is defined over the constant field.

Lemma 2.5. *Suppose that X has finite stabilizer and every W_m has dense small points. Then*

$$X = a + \mathrm{tr}(Y_{\overline{K}}) \tag{9}$$

for some $a \in G(\overline{K})$ and some closed integral subvariety $Y \subset C$ defined over k. The stabilizer of Y is finite.

Proof. Put $d = \dim X$, and fix $m > \dim G$, with $m \geq 2$. By Lemma 2.4, the stabilizers of W_m and W_{m+1} are finite. Geometric Bogomolov and the universal property of the trace give, for $j = m, m+1$,

$$W_j = t_j + \mathrm{tr}^{j-1}(V_j),$$

where t_j is torsion and $V_j \subset C^{j-1}$ is closed and defined over k. Indeed, the homomorphism in the Bogomolov description factors through the trace by a homomorphism with finite kernel; its image of the given closed subvariety is therefore closed.

Since diagonal tuples give $0 \in W_j$, we have $t_j \in H^{j-1}$. A lift of t_j to $C^{j-1}(\overline{K})$ is torsion, because the trace has finite kernel. All torsion points of the constant semiabelian variety C^{j-1} are defined over k. Translating V_j by such a lift removes t_j.

Set

$$Z_j = \left((\mathrm{tr}^{j-1})^{-1}(W_j)\right)_{\mathrm{red}}.$$

The variety Z_j is defined over k. On geometric points it is the union of the translates of V_j by the finite group $\ker(\mathrm{tr}^{j-1})(\overline{K})$, whose points are torsion k-points. This determines its reduced closed subscheme. No assertion about descent of the infinitesimal part of the kernel is needed.

Projection to any coordinate shows that $X - X \subset H$. Let $p : W_{m+1} \to W_m$ forget the last coordinate. It is dominant and its generic fiber has dimension d, since $\dim W_j = jd$. Choose a nonempty open $U \subset W_m$ contained in $\alpha_m(X^m)$ such that the fibers of α_m over U are finite and the fibers of p over U have dimension d.

The inverse image of U in Z_m is nonempty open. Since $Z_m(k)$ is dense after extension to $\overline{K}$, choose $w' \in Z_m(k)$ in this open and put $w = \mathrm{tr}^{m-1}(w')$. Choose a tuple above w under α_m, and denote its first coordinate by a. The projection $Z_{m+1} \to Z_m$ is defined over k. Let $Z_{w'}$ be the reduction of its fiber over w', viewed as a closed subvariety of the last copy of C. On geometric points it is the inverse image under tr of the fiber $p^{-1}(w)$. It follows that $\dim Z_{w'} = d$.

Varying the last point of the chosen tuple shows that $\mathrm{tr}^{-1}(X - a)_{\mathrm{red}} \subset Z_{w'}$. Choose an irreducible component Y of this inverse image which maps onto $X - a$. Such a component exists because tr is finite and surjective onto H. It has dimension d, so it is an irreducible component of $Z_{w'}$. This fiber is defined over the algebraically closed field k, and hence so is Y. We obtain (9). Finally, $\mathrm{tr}(\mathrm{Stab}(Y)) \subset \mathrm{Stab}(X)$, so the stabilizer of Y is finite. □

2.5. **Bounded approximation.** For a constant subvariety with finite stabilizer, small differences force the original points to be small on a nonempty open subset. The usual height comparison for a generically finite map has an additive constant. Here all models are defined over k, so intersection degrees give the comparison without an additive error.

Lemma 2.6. *Let $Y \subset C$ be a closed integral subvariety over k with finite stabilizer, and let $\widehat{h}_C$ be a standard canonical height on $C_{\overline{K}}$. There are an integer $r \geq 2$, a nonempty open subset $U \subset Y^r$ defined over k, and a real number $c > 0$ such that*

$$\sum_{i=1}^{r} \widehat{h}_C(y_i) \leq c \sum_{i=2}^{r} \widehat{h}_C(y_i - y_1) \qquad \big((y_1, \ldots, y_r) \in U(\overline{K})\big). \tag{10}$$

Proof. If Y is a point, then it is a k-point and both sides vanish. Otherwise we may choose the height bundles and compactifications over k, by height comparison. Take $r > \dim C$, and take the projective closure of the graph of the difference map $Y^r \to C^{r-1}$. Let p be its projection to a compactification of Y^r, and let f be its projection to the compactified difference image. Let L_1 and L_2 be the ample line bundles induced by the standard product bundles on these two compactifications. Their constant-model heights on the open parts are the left-hand sum and the right-hand sum in (10), before multiplication by c.

Lemma 2.4 shows that f is generically finite. Thus f^*L_2 is big. There are positive integers b, c and a nonzero section over k of

$$(f^*L_2)^{\otimes bc} \otimes (p^*L_1)^{\otimes(-b)}.$$

Remove the zero locus of this section from the open graph, obtaining a nonempty open $U \subset Y^r$. A constant effective divisor has nonnegative geometric height away from its support: pull back its section to a normal projective model of the point's field of definition, extend at codimension-one points, and take the polarized degree of the resulting effective divisor.

Additivity and functoriality on constant models therefore give

$$bc\,\widehat{h}_{L_2}(f(z)) - b\,\widehat{h}_{L_1}(p(z)) \geq 0 \qquad (z \in U(\overline{K})).$$

The model heights agree with the canonical heights because the multiplication identities already hold on the constant models. Division by b proves (10). □

We now combine the preceding arguments. The common centers first make the difference images small. After recovering a constant translate, the height inequality shows that its translating point can itself be approximated by Γ.

Proposition 2.7. *Let $X \subset G$ be closed and integral with finite stabilizer. The following conditions are equivalent.*

(1) There is a real number $R \geq 0$ such that, for every $\epsilon > 0$, the set

$$\{x \in X(\overline{K}) : \widehat{h}(x) \leq R,\ d_\Gamma(x) < \epsilon\}$$

is Zariski dense in X.

(2) There are $\gamma \in \Gamma$ and a closed integral k-subvariety $Y \subset C$ such that

$$X = \gamma + \mathrm{tr}(Y_{\overline{K}}). \tag{11}$$

Proof. Assume the first condition. For each integer $n \geq 1$, choose a dense set of points x with $\widehat{h}(x) \leq R$, together with points $\gamma \in \Gamma$, such that $\rho(x-\gamma) < 1/n$. The triangle inequality gives

$$\rho(\gamma) \leq R + \sqrt{R} + 1.$$

Modulo the kernel of ρ, these centers lie in a bounded subset of the finite-dimensional normed space generated by Γ. Cover this subset by finitely many balls of radius $1/n$ whose centers come from Γ. By irreducibility, one ball corresponds to a dense subset $E_n \subset X(\overline{K})$. Thus there is $\lambda_n \in \Gamma$ such that, with $b_n = 2/n$,

$$\rho(x - \lambda_n) \leq b_n \qquad (x \in E_n). \tag{12}$$

For each fixed m, every coordinate of $\alpha_m(E_n^m)$ has ρ-value at most $2b_n$. By (1), its product height is at most

$$(m-1)(2b_n + 4b_n^2).$$

Its image is dense in W_m. Letting $n \to \infty$, we see that every W_m has dense small points. Lemma 2.5 gives $X = a + \mathrm{tr}(Y_{\overline{K}})$, where $Y \subset C$ is defined over k and has finite stabilizer.

The inverse image of $E_n - a$ in Y is dense because $Y \to X - a$ is finite and surjective. Choose a tuple from its r-th power in the open subset of Lemma 2.6, and write $x_i = a + \mathrm{tr}(y_i) \in E_n$. Equation (12) gives $\rho(x_i - x_1) \leq 2b_n$. Lemma 2.1 for the finite-kernel homomorphism tr implies

$$\widehat{h}_C(y_i - y_1) \longrightarrow 0.$$

Equation (10) then gives $\widehat{h}_C(y_1) \to 0$, and upper height comparison gives $\rho(\mathrm{tr}(y_1)) \to 0$. Consequently

$$\rho(a - \lambda_n) \leq \rho(x_1 - \lambda_n) + \rho(\mathrm{tr}(y_1)) \longrightarrow 0.$$

Lemma 2.3 writes $a = \gamma + \mathrm{tr}(c)$ with $\gamma \in \Gamma$ and $c \in C(k)$. Replacing Y by $Y + c$ proves the second condition.

Conversely, suppose that (11) holds. The set $\gamma + \mathrm{tr}(Y(k))$ is dense in X, and its points differ from γ by points of height zero. The triangle inequality in both directions gives

$$\rho(\gamma + \mathrm{tr}(y)) = \rho(\gamma) \qquad (y \in Y(k)).$$

Their heights are therefore bounded by $\rho(\gamma)+\rho(\gamma)^2$, while their distance to Γ is zero. □

3. A consequence of finitely generated Mordell–Lang

Let $k_0 \subset k_1$ be algebraically closed fields. In this section we explain how a projection to a constant semiabelian variety can force a subvariety with dense finitely generated intersection to come from the constant field. The hypothesis needed for this purpose is that no positive-dimensional subgroup in the other factor stabilizes the subvariety. We first specify the translating point in positive-characteristic Mordell–Lang, and then apply the resulting description to the product.

We shall use the following standard descent fact: a connected reduced subgroup of a semiabelian variety over k_0, after extension to k_1, is defined over k_0. Homomorphisms between semiabelian varieties over k_0 likewise do not change after extension of algebraically closed fields; see [BBP16, Fact 2.4(iii) and §2.2.2]. The first assertion can be seen from prime-to-characteristic torsion. These torsion points are dense in the subgroup and already belong to the constant field, since the corresponding torsion schemes are finite over k_0. The second assertion follows by applying the first to the graph of the homomorphism.

We also need to lift division points. If $h : H_0 \to H_1$ is a surjective homomorphism of semiabelian varieties over k_1, and $\Lambda_0 \subset H_0(k_1)$ is a subgroup, then

$$h(\Lambda_0^{\mathrm{div}}) = h(\Lambda_0)^{\mathrm{div}}. \tag{13}$$

To verify this, first note that every torsion point of H_1 has a torsion lift. Indeed, factor h through the quotient by its connected reduced kernel, followed by an isogeny. For the first map, a lift of an n-torsion point can be corrected by a point of the divisible kernel to become n-torsion. For the isogeny, every lift of a torsion point is torsion. Now suppose that $[n]\alpha = h(\ell)$, where $\ell \in \Lambda_0$. Choose $b_0 \in H_0(k_1)$ with $[n]b_0 = \ell$, and lift the torsion point $h(b_0)-\alpha$ to a torsion point $t \in H_0(k_1)$. Then $b = b_0 - t$ satisfies $h(b) = \alpha$, and some positive multiple of b belongs to Λ_0. This proves the nontrivial inclusion in (13).

The finite-field step below specifies the translating point. We retain the Frobenius orbit description in its stable module and impose membership in the original division group on the exponent tuples.

Lemma 3.1. *Let Ω be an algebraically closed field of positive characteristic, let H be a semiabelian variety defined over a finite subfield of Ω, and let $\Delta \subset H(\Omega)$ be finitely generated. If an integral closed subvariety $V \subset H_\Omega$ has finite stabilizer and $V(\Omega)\cap\Delta$ is Zariski dense, then*

$$V = \delta + (V_1)_\Omega$$

for some $\delta \in \Delta^{\mathrm{div}}$ and some integral closed subvariety $V_1 \subset H$ defined over a finite field.

Proof. Choose a finite field $\mathbb{F}_q$ over which H is defined and its torus splits, and let F be the corresponding Frobenius. Frobenius on the abelian quotient is annihilated by a squarefree monic polynomial $P_A \in \mathbb{Z}[U]$, whose complex roots have absolute value $q^{1/2}$. On the torus, $F = [q]$. Hence $(U-q)P_A(U)$ annihilates F on H: the image of $P_A(F)$ is in the torus and is killed by $F-q$. This polynomial is squarefree because the two root absolute values differ. The absent factor is omitted if either part is zero. In particular, F is integral, and its action on every rational representation annihilated by this polynomial is semisimple with eigenvalues of absolute value greater than one; see [CGSZ21, §2.1] for these Frobenius properties.

The group $\widetilde{\Delta} = \mathbb{Z}[F]\Delta$ is therefore finitely generated as an abelian group. The theorem of Moosa–Scanlon [MS04, Theorem 7.8] expresses $V(\Omega) \cap \widetilde{\Delta}$ as a finite union of sets $S + B$, where B is a subgroup and S is a translate of a sum of finitely many Frobenius orbits. Their theorem is stated over a finitely generated regular extension of a finite field. Choose a field containing the finite data and enlarge its finite constant field; replacing q by the size of that field and F by the corresponding power gives that setting, as in [MS04, Remark 7.11]. The module remains stable and finitely generated. Their formulation in terms of Frobenius cycles gives the stated orbit form by [MS04, Lemma 7.1]: if $(F^e - 1)a = b$, then

$$b + F^e b + \cdots + F^{en} b = -a + F^{e(n+1)} a.$$

Such an a exists because $F^e - 1$ is an isogeny. Orbit base points may thus lie in $H(\Omega)$, which causes no difficulty below.

If B is infinite, the closure of $S + B$ is invariant under the positive-dimensional group $\overline{B}$. It is a proper closed subset of V, since V has finite stabilizer. A finite B can be absorbed into finitely many translates of S. Density and irreducibility therefore give one set

$$S = \left\{ a_0 + \sum_{i=1}^{r} F^{e_i n_i} a_i : n_i \geq 0 \right\} \subset V(\Omega), \qquad \overline{S \cap \Delta} = V, \tag{14}$$

where each e_i is positive. If $r = 0$, the assertion is immediate. Assume $r > 0$, and write $s(n)$ for the point indexed by $n = (n_1, \ldots, n_r)$.

We first determine which exponent tuples give points in the division hull. Let M be the $\mathbb{Z}[F]$-module generated by Δ and $a_0, \ldots, a_r$, and put $M_{\mathbb{Q}} = M \otimes_{\mathbb{Z}} \mathbb{Q}$. This is finite-dimensional. Write $\Delta_{\mathbb{Q}}$ for the rational span of the image of Δ. For $x \in M$,

$$x \in \Delta^{\mathrm{div}} \quad \Longleftrightarrow \quad x \otimes 1 \in \Delta_{\mathbb{Q}}.$$

For the nontrivial implication, clear rational denominators and then kill the resulting torsion element by a further integer multiple.

Choose rational linear forms whose common kernel is $\Delta_{\mathbb{Q}}$, and diagonalize F over a number field containing its eigenvalues λ_j. The condition $s(n) \otimes 1 \in \Delta_{\mathbb{Q}}$ becomes a finite system

$$c_{\ell,0} + \sum_{i,j} c_{\ell,i,j} \lambda_j^{e_i n_i} = 0.$$

All coefficients lie in that number field. Each function $n \mapsto \lambda_j^{e_i n_i}$ is a character of $\mathbb{Z}^r$. By Laurent's theorem on constant-coefficient sums of characters [Lau87, Theorem 1(i)], the solution set in $\mathbb{Z}^r$ is a finite union of cosets of subgroups. Thus the nonnegative exponent tuples giving points in Δ^{div} form a finite union of sets

$$\mathcal{E} = (c + L) \cap \mathbb{Z}^r_{\geq 0}, \qquad c \in \mathbb{Z}^r, \quad L \subset \mathbb{Z}^r.$$

Their images contain $S \cap \Delta$ and lie in V. Choose one whose image is dense in V.

The next observation separates bounded coordinates from directions in which all remaining coordinates can increase. Dickson's lemma says that $\mathcal{E}$ has finitely many minimal elements for the coordinatewise order. One elementary proof is that every infinite sequence in $\mathbb{Z}^r_{\geq 0}$ has an infinite subsequence nondecreasing in each coordinate: successively retain a constant subsequence when a coordinate repeats infinitely often, and otherwise a strictly increasing one. An infinite set of minimal elements would contradict this property. Every $n \in \mathcal{E}$ dominates a minimal element b, and $n - b \in L \cap \mathbb{Z}^r_{\geq 0}$.

Let J be the coordinates positive on some vector of $L\cap\mathbb{Z}^r_{\geq 0}$, and let I be the complementary coordinates. Each coordinate in I takes only finitely many values on $\mathcal{E}$, namely values at its finitely many minimal elements. Sum one nonnegative lattice vector positive in each coordinate of J. This gives $v \in L\cap\mathbb{Z}^r_{\geq 0}$ with $v_i = 0$ for $i \in I$ and $v_j > 0$ for $j \in J$.

Fix values of the coordinates in I so that the remaining subset $\mathcal{E}_b \subset \mathcal{E}$ still has dense image in V, and choose $b \in \mathcal{E}_b$. Put

$$\delta = a_0 + \sum_{i\in I} F^{e_i b_i} a_i, \qquad \beta_j = F^{e_j b_j} a_j \quad (j \in J).$$

For every integer $n \geq 0$, the tuple $b + nv$ belongs to $\mathcal{E}_b$. Hence

$$x_n = \delta + \sum_{j\in J} (F^{e_j v_j})^n \beta_j \in \Delta^{\mathrm{div}}.$$

For each $j \in J$, choose an integer polynomial Q_j annihilating $F^{e_j v_j}$ with $Q_j(1) \neq 0$; its eigenvalues have absolute value greater than one. Write $Q(U) = \prod_{j\in J} Q_j(U) = \sum_{n=0}^{N} q_n U^n$. Applying this polynomial in the shift operator to the sequence gives

$$\sum_{n=0}^{N} q_n x_n = Q(1)\delta \in \Delta^{\mathrm{div}}.$$

The full division hull is saturated under multiplication by every nonzero integer, so $\delta \in \Delta^{\mathrm{div}}$. If J is empty, this follows directly from $x_0 = \delta$.

Finally, keep only the coordinates in I fixed and let all the other original exponents vary independently. The resulting subset

$$S' = \left\{ \delta + \sum_{j\in J} F^{e_j n_j} a_j : n_j \geq 0 \right\}$$

lies in $S \subset V$ and contains the dense image of $\mathcal{E}_b$. Thus $\overline{S'} = V$. For $j \in J$, set

$$W_j = \overline{\{F^{e_j n} a_j : n \geq 0\}}.$$

The descending sequence of tail closures stabilizes. Since F^{e_j} is finite and bijective on geometric points, the stable tail closure is invariant and contains every preceding orbit point. Consequently $F^{e_j}(W_j) = W_j$.

This invariance implies descent to $\mathbb{F}_{q^{e_j}}$. Indeed, embed H locally closed in projective space over $\mathbb{F}_q$ and take the projective closure of W_j. Each graded piece of its homogeneous ideal is invariant under coefficientwise q^{e_j}-power Frobenius. The ratios of its Plücker coordinates to a nonzero coordinate are therefore fixed by that Frobenius and lie in $\mathbb{F}_{q^{e_j}}$. Each graded piece, and hence the ideal, descends to this field.

The product of the orbit sets is dense in the product of their closures. Taking the image under addition and then its closure gives

$$V = \delta + V_1, \qquad V_1 = \overline{\sum_{j\in J} W_j}.$$

For an empty J, take $V_1 = \{0\}$. The variety V_1 is defined over a finite field and is integral because it is a translate of V. □

Lemma 3.2. *Assume that* $\operatorname{char} k_1 = p > 0$. *Let* H/k_1 *be a semiabelian variety, let* $\Lambda \subset H(k_1)$ *be finitely generated, and let* $Z \subset H$ *be an integral closed subvariety such that* $Z(k_1) \cap \Lambda$ *is Zariski dense in* Z. *Then there exist a semiabelian subvariety* $H_0 \subset H$, *a semiabelian variety* H_1 *and an integral closed subvariety* $V_1 \subset H_1$ *defined over a finite field, a surjective homomorphism* $h : H_0 \to (H_1)_{k_1}$, *and* $\lambda \in \Lambda^{\mathrm{div}}$ *such that*

$$Z = \lambda + h^{-1}((V_1)_{k_1}).$$

Proof. Hrushovski's theorem [Hru96, Theorem 1.1], applied with constant field $\overline{\mathbb{F}}_p$, gives

$$Z = a + h^{-1}(V_0), \tag{15}$$

where $H_0 \subset H$ is semiabelian, $h : H_0 \to (H_1)_{k_1}$ is surjective, and H_1 and $V_0 \subset H_1$ are defined over a finite field. We suppress scalar-extension subscripts in the rest of this proof. To obtain this form from the rational-homomorphism formulation, extend the rational group homomorphism to a morphism and replace its target by its image. This image is defined over $\overline{\mathbb{F}}_p$ by subgroup descent. Intersect the original target subvariety with the image; it is integral because it is the image of $Z - a$. The resulting finite data descend to a finite field. The point a is unrestricted; the remaining argument places a translator in Λ^{div}.

The connected reduced stabilizer of V_0 is defined over a finite field. Replacing H_1 by its quotient by this stabilizer, and composing h with the quotient map, we may suppose that V_0 has finite stabilizer. This does not change (15): the subvariety V_0 is the full inverse image of its closed image in the quotient.

Choose $x_0 \in Z(k_1) \cap \Lambda$, and set

$$\Lambda_0 = \Lambda \cap H_0(k_1), \qquad \Delta = h(\Lambda_0), \qquad V = h(Z - x_0).$$

The variety V is a translate of V_0, so it is closed and integral and has finite stabilizer. Moreover,

$$Z - x_0 = h^{-1}(V),$$

and $V(k_1) \cap \Delta$ is dense in V. The groups Λ_0 and Δ are finitely generated.

Apply Lemma 3.1 to V and Δ. It gives

$$V = \delta + V_1, \qquad \delta \in \Delta^{\mathrm{div}},$$

where V_1 is defined over a finite field. By (13), choose $b \in \Lambda_0^{\mathrm{div}}$ with $h(b) = \delta$. Then $\lambda = x_0 + b \in \Lambda^{\mathrm{div}}$ and

$$Z = x_0 + h^{-1}(\delta + V_1) = \lambda + h^{-1}(V_1).$$

□

We now use the projection to the constant factor. The quotient supplied by Mordell–Lang need not have a complementary semiabelian subvariety. Instead, we combine it with the projection. The two maps together will have finite kernel, which is sufficient to recover the required constant variety by an isogeny.

Proposition 3.3. *Let* P/k_0 *and* G/k_1 *be semiabelian varieties, let* $\Lambda \subset (P_{k_1} \times G)(k_1)$ *be finitely generated, and let* $Z \subset P_{k_1} \times G$ *be an integral closed subvariety such that*

$$\overline{Z(k_1) \cap \Lambda} = Z, \qquad \dim\big(\operatorname{Stab}(Z) \cap (\{0\} \times G)\big) = 0. \tag{16}$$

Then there exist $\lambda \in \Lambda^{\mathrm{div}}$, *a semiabelian variety* J/k_0, *an integral closed subvariety* $Y \subset J$, *and a homomorphism* $j : J_{k_1} \to P_{k_1} \times G$ *with finite kernel such that*

(1)
$$Z = \lambda + j(Y_{k_1});$$
(2) the homomorphism $\mathrm{pr}_P \circ j : J_{k_1} \to P_{k_1}$ *is defined over* k_0.

If $\mathrm{char}\, k_1 = 0$, *one may take* $\lambda \in \Lambda$ *and* $Y = J$.

Proof. Suppose first that $\mathrm{char}\, k_1 > 0$. Apply Lemma 3.2 to $Z \subset P_{k_1} \times G$, and retain its notation. The finite field of definition of H_1 and V_1 is contained in k_0. Consider
$$f = (\mathrm{pr}_P, h) : H_0 \longrightarrow P_{k_1} \times (H_1)_{k_1}.$$
Every geometric point of $\ker h$ stabilizes $h^{-1}(V_1) = Z - \lambda$. A point in $\ker f$ also has zero first coordinate. Consequently,
$$(\ker f)_{\mathrm{red}} \subset \mathrm{Stab}(Z) \cap (\{0\} \times G),$$
so (16) implies that $\ker f$ is zero-dimensional and hence finite. Let J_{k_1} be the image of f. It is a connected reduced subgroup of the constant semiabelian variety $P_{k_1} \times (H_1)_{k_1}$, and hence descends to a semiabelian subvariety $J \subset P \times H_1$ over k_0. The resulting map $f : H_0 \to J_{k_1}$ is an isogeny.

Choose a positive integer m that annihilates the finite group scheme $\ker f$. Multiplication by m on H_0 factors through f, giving an isogeny $j : J_{k_1} \to H_0$ such that
$$f \circ j = [m]_J, \qquad j \circ f = [m]_{H_0}.$$
Using the scheme-theoretic kernel here includes the case of an inseparable isogeny. Taking the two components of the first equality gives
$$h \circ j = [m]\,\mathrm{pr}_{H_1}, \qquad \mathrm{pr}_P \circ j = [m]\,\mathrm{pr}_P, \tag{17}$$
where the projections on the right are restricted to J. Both are defined over k_0.

Let
$$Y^* = \big(([m]\,\mathrm{pr}_{H_1})^{-1}(V_1)\big)_{\mathrm{red}} \subset J.$$
By (17) and the surjectivity of j,
$$j(Y^*_{k_1}) = h^{-1}(V_1).$$
The map j is finite, so its images of closed subvarieties are closed. Since the target is integral, some irreducible component Y of Y^* maps onto it. The field k_0 is algebraically closed, so this component is defined over k_0. Composing j with the inclusion $H_0 \subset P_{k_1} \times G$ gives both assertions of the proposition.

If $\mathrm{char}\, k_1 = 0$, use the finitely generated case of Mordell–Lang over $\mathbb{C}$, proved in [McQ95, Introduction, Conjecture 1, and §3]. Here is the passage to the present field. Descend the ambient semiabelian variety, Z, the generators of Λ, and a point $\lambda \in Z(k_1) \cap \Lambda$ to a finitely generated field $E \subset k_1$ over $\mathbb{Q}$. Embed E in $\mathbb{C}$. The intersection points are E-rational; expansion of coefficients in an E-linearly independent family shows that their density persists after extension to $\mathbb{C}$. Mordell–Lang makes $Z_{\mathbb{C}}$ a coset. Since it contains λ, the translate $Z_{\mathbb{C}} - \lambda$ is a subgroup. The group identities descend faithfully to E and then extend to k_1. Thus $Z = \lambda + H_0$, with $\lambda \in \Lambda$ and H_0 a semiabelian subvariety. The restriction $H_0 \to P_{k_1}$ of the first projection has finite kernel by (16), and its image descends to a semiabelian subvariety $J \subset P$. The same factorization of multiplication through this isogeny gives $j : J_{k_1} \to H_0$. Taking $Y = J$ proves the result. □

4. Bounded replacement

We now replace dense approximation by bounded dense approximation after one extension of the constant field. The original points can have unbounded height. The replacement points will lie on the same subvariety and remain close to the same division group.

For an algebraically closed extension k'/k, form $\overline{K} \otimes_k k'$ using abstract copies of the two fields. This tensor product is a domain. Embed its fraction field in an algebraic closure $\overline{K'}$ of the compositum $K' = Kk'$. These embeddings are linearly disjoint over k, and $\overline{K} \cap k' = k$. We call this an independent extension of constants. Equip K'/k' with the scalar extension of the fixed polarization.

Lemma 4.1. *For every $a \in G(\overline{K})$, we have*

$$\widehat{h}_{K'/k'}(a) = \widehat{h}_{K/k}(a), \qquad \rho_{K'/k'}(a) = \rho_{K/k}(a).$$

Proof. Choose a finite extension E/K over which a is defined. Since k is algebraically closed, the extension E/k is regular. In particular, $E \otimes_K K'$ is a domain: it is a localization of $E \otimes_k k'$. It is finite over the field K', and is therefore a field of degree $[E : K]$.

The intersection numbers defining the model height are preserved by extension of the constant field. The preceding degree equality shows that the normalization of the height is preserved as well. Thus the model heights of every $[n]a$ agree before and after extension of constants. Taking the limits defining $\widehat{h}_D$ and $\widehat{h}_L$ proves the assertion. This argument does not require E/K to be separable. □

Proposition 4.2. *Suppose that X has finite stabilizer and that $\{x \in X(\overline{K}) : d_\Gamma(x) < \epsilon\}$ is Zariski dense for every $\epsilon > 0$. There exist an independent algebraically closed extension k'/k and a real number $R \geq 0$ such that, for every $\epsilon > 0$,*

$$\{y \in X(\overline{K'}) : \widehat{h}_{K'/k'}(y) \leq R,\ d_\Gamma(y) < \epsilon\}$$

is Zariski dense in $X_{\overline{K'}}$. Here Γ is the original subgroup of $G(\overline{K})$, and the distance in the display is computed with $\rho_{K'/k'}$.

The extension and the height bound are chosen once, independently of ϵ. The proof occupies the rest of this section. We first parametrize the bounded remainders by equations over the constants, then use finitely many independent field embeddings to apply Proposition 3.3.

We call a sequence in an integral variety *generic* if every proper closed subvariety contains only finitely many of its terms. Every sequence has a subsequence generic in an integral closed subvariety: choose a minimal closed subset containing infinitely many terms, and then a component containing infinitely many terms. Minimality excludes infinitely many terms in any proper closed subset. A sequence generic in the whole variety remains so after passing to a subsequence.

4.1. **Bounded remainders.** Choose generators $P_1, \dots, P_s$ of Γ_0. By our convention they belong to $G(K)$. Suppose that

$$x_i = \gamma_i + e_i \in X(\overline{K}), \qquad \gamma_i \in \Gamma, \qquad \rho(e_i) \longrightarrow 0.$$

In the rational vector space $G(\overline{K})/G(\overline{K})_{\mathrm{tors}}$, write $\gamma_i = \sum_j q_{ij} P_j$, with $q_{ij} \in \mathbb{Q}$, and put

$$\eta_i = \sum_j \lfloor q_{ij} \rfloor P_j \in \Gamma_0, \qquad z_i = x_i - \eta_i.$$

The coefficients of $\gamma_i - \eta_i$ lie in $[0,1)$. The seminorm inequalities therefore give

$$\rho(\gamma_i - \eta_i) \le \sum_j \rho(P_j), \qquad \sup_i \rho(z_i) < \infty, \qquad d_\Gamma(z_i) = d_\Gamma(x_i) \longrightarrow 0.$$

The height comparison in §2 implies that the z_i have bounded projective height in the fixed equivariant compactification of G. The centers η_i need not have bounded height. Our purpose is to describe enough of the possible centers by equations with parameters in k.

4.2. **Linear equations of points of bounded height.** Let $R = k(t_1, \dots, t_d)$, where $t_1, \dots, t_d$ are algebraically independent over k and $d \ge 1$. In this subsection h_R is the geometric projective height associated with $\mathcal{O}_{\mathbb{P}^d_k}(1)$, normalized by degrees of finite field extensions. Thus a rational point with primitive polynomial coordinates has height equal to their maximum total degree.

For an integer $n \ge 1$ and a point $P \in \mathbb{P}^{n-1}(\overline{R})$, let $S_R(P) \subset R^n$ be the smallest R-linear subspace whose scalar extension contains the line represented by P. Equivalently, it is the common kernel of the R-linear forms vanishing at P. We use this subspace because every linear equation of P vanishes on all of $S_R(P)$. The following lemma shows that bounded height puts these subspaces in a family of finite type over k.

Lemma 4.3. *Let $C \ge 0$ be a real number. If $h_R(P) \le C$, then $S_R(P)$ has a basis of vectors in $k[t_1, \dots, t_d]^n$ whose entries have total degree at most $\lfloor nC \rfloor$.*

Proof. Suppose first that $\mathrm{char}(k) = p > 0$ and that P is defined over the perfect closure $R^{\mathrm{perf}} = \bigcup_{e \ge 0} R^{1/p^e}$. Choose $Q = p^e$ such that the coordinatewise Q-th power of P is R-rational. Write

$$P^{(Q)} = [f_1 : \cdots : f_n], \qquad f_j \in k[t_1, \dots, t_d],$$

with no common nonconstant factor. If $D = \max_j \deg f_j$, then

$$D = h_R(P^{(Q)}) = Q h_R(P).$$

Indeed, every finite prime divisor contributes zero to the height of these primitive coordinates, and the hyperplane at infinity contributes their maximum degree.

Since k is perfect, each f_j has a unique expression

$$f_j = \sum_{\boldsymbol{b} \in \{0, \dots, Q-1\}^d} t^{\boldsymbol{b}} g_{j,\boldsymbol{b}}^Q, \qquad g_{j,\boldsymbol{b}} \in k[t_1, \dots, t_d].$$

Here $t^{\boldsymbol{b}} = \prod_{a=1}^d t_a^{b_a}$ and $|\boldsymbol{b}| = \sum_a b_a$. For every nonzero summand,

$$Q \deg g_{j,\boldsymbol{b}} + |\boldsymbol{b}| \le D, \qquad \deg g_{j,\boldsymbol{b}} \le h_R(P).$$

Taking Q-th roots gives

$$(f_1^{1/Q}, \dots, f_n^{1/Q}) = \sum_{\boldsymbol{b}} t^{\boldsymbol{b}/Q} (g_{1,\boldsymbol{b}}, \dots, g_{n,\boldsymbol{b}}).$$

The elements $t^{\boldsymbol{b}/Q}$ form an R-basis of $R^{1/Q}$. An R-linear form vanishes on the left-hand side if and only if it vanishes on every coefficient vector on the right-hand side. Those coefficient vectors therefore span $S_R(P)$ and have the required degree bound. In characteristic zero, the same assertion for an R-rational point follows directly from its primitive polynomial coordinates.

For an arbitrary P, let $V \subset \overline{R}^n$ be the span of its Galois conjugate lines. Choose conjugates $P^{\sigma_1}, \dots, P^{\sigma_r}$ spanning V, where $r \le n$ and $\sigma_j \in \mathrm{Gal}(\overline{R}/R)$. Write $h_R(V)$ for the height of the

Plücker point of V, that is, the projective point given by the maximal minors of a basis matrix. At each place, the ultrametric inequality bounds every determinant by the product of the maximum coordinate norms of its rows. Applied to the Plücker coordinates, this gives

$$h_R(V) \le \sum_{j=1}^{r} h_R(P^{\sigma_j}) = r\,h_R(P) \le nC.$$

The Plücker ratios of V are fixed by $\mathrm{Gal}(\overline{R}/R)$. They consequently belong to R^{perf} in positive characteristic and to R in characteristic zero. Choose a row-echelon basis using a nonzero Plücker coordinate. After multiplication by the chosen nonzero Plücker coordinate, each basis row has coordinates consisting of that coordinate, other signed Plücker coordinates, and zeros. Its projective height is therefore at most $h_R(V)$.

Apply the preceding argument to each row of this basis. Its smallest R-linear container is spanned by polynomial vectors of degree at most $\lfloor nC \rfloor$. The sum of these containers is $S_R(P)$: an R-linear form vanishes at P if and only if it vanishes on every Galois conjugate, hence on V. Selecting a basis from the resulting polynomial vectors proves the lemma. □

4.3. **Equations over a general function field.** We next apply the lemma when the field of definition K need not be rational, and when the equations are polynomial rather than linear. Two elementary changes of coordinates suffice.

Use the polarized model (S, M) fixed in the introduction, with $d = \dim S$. Projective Noether normalization, applied to a sufficiently high power of M, gives a finite morphism

$$\nu : S \longrightarrow \mathbb{P}^d_k, \qquad \nu^*\mathcal{O}(1) \simeq M^{\otimes a}$$

for some positive integer a. Set $R = k(\mathbb{P}^d) \subset K$, let $e = [K : R]$, and choose an R-basis $\beta_1, \dots, \beta_e$ of K. Write h_M for the ordinary projective height associated with the polarized model (S, M). For a projective point over an algebraic extension of K, the normalized heights satisfy

$$h_R(P) = \frac{a^{d-1}}{e}\, h_M(P).$$

To see this, define P over a finite extension E/K and normalize the models in E. The $d-1$ base polarization classes contribute a^{d-1}, while $[E : R] = e[E : K]$. This calculation also applies to inseparable extensions.

Fix a positive integer m. Let s be the number of degree-m monomials in n variables. Define

$$\Psi : \mathbb{P}^{n-1}_K \longrightarrow \mathbb{P}^{es-1}_K, \qquad [x] \longmapsto [\beta_b x^\alpha]_{b,\alpha}, \tag{18}$$

where $x = (x_1, \dots, x_n)$ is a nonzero homogeneous coordinate vector, $1 \le b \le e$, and α runs through the multi-indices of total degree m. This is the Veronese embedding followed by a fixed linear embedding. The maximum norms give

$$h_R(\Psi(P)) = m\,h_R(P) + h_R([\beta_1 : \cdots : \beta_e]). \tag{19}$$

This embedding converts every degree-m equation over K into a linear equation over R. Indeed, for

$$f(X) = \sum_\alpha a_\alpha X^\alpha, \qquad a_\alpha = \sum_{b=1}^{e} \beta_b c_{b,\alpha}, \quad c_{b,\alpha} \in R,$$

put $\ell_f(Y) = \sum_{b,\alpha} c_{b,\alpha} Y_{b,\alpha}$. For compatible homogeneous coordinates,

$$\ell_f(\Psi(P)) = f(P).$$

If $f(P)=0$, the linear form ℓ_f thus vanishes on $S_R(\Psi(P))$. The use of the basis β_b avoids any separability assumption on K/R.

4.4. **A family defined over the constant field.** Fix a projective embedding $\overline{G}\hookrightarrow\mathbb{P}^{n-1}_K$ of the standard equivariant compactification. Suppose that

$$\eta_i\in G(K),\qquad z_i\in G(\overline{K}),\qquad x_i=\eta_i+z_i\in X(\overline{K}),$$

and that the z_i have bounded projective height. The preceding construction has the following consequence.

Proposition 4.4. *After passing to a subsequence, there are an integral k-variety V, points $t_i\in V(k)$, and a closed subset $\mathcal{R}\subset G\times V_K$ such that*

$$(\eta_i,t_i)\in\mathcal{R},\qquad z_i+\mathcal{R}_{t_i}\subset X \tag{20}$$

for every i.

The second inclusion is the essential assertion. It says that, with z_i fixed, every center in the fiber $\mathcal{R}_{t_i}$ still produces a point of X. Notice that the parameter variety V is defined over k, whereas the incidence $\mathcal{R}$ is defined over K.

Proof. Translation identifies the family of translates of $\overline{X}$ with $\overline{X}\times G$, so it is flat and projective over G. Its ideal sheaf in $\mathbb{P}^{n-1}\times G$ is flat over G. The locus where a fiber ideal sheaf is m-regular is open by upper semicontinuity of fiber cohomology [Stacks, Tag 0BDN]. These opens increase with m [Stacks, Tag 08A6] and cover G by Serre vanishing. Since G is quasi-compact, one positive integer m works for every fiber.

For $\eta\in G(K)$, let $\mathcal{I}_\eta$ be the ideal sheaf of $\overline{X-\eta}$ in $\mathbb{P}^{n-1}_K$. By m-regularity, $\mathcal{I}_\eta(m)$ is globally generated [Stacks, Tag 08A8]. Its global sections are the degree-m forms over K vanishing on $\overline{X-\eta}$. Twisting the evaluation surjection by $\mathcal{O}(-m)$ shows that these forms generate $\mathcal{I}_\eta$ as an ideal sheaf. Thus their common zero scheme is $\overline{X-\eta}$.

Use this m in (18). Set

$$S_i=S_R(\Psi(z_i)),\qquad Z_i=G\cap\Psi^{-1}(\mathbb{P}(S_i)_K).$$

Every degree-m equation f of $\overline{X-\eta_i}$ vanishes at z_i. Its associated R-linear form ℓ_f vanishes on S_i. Hence

$$z_i\in Z_i\subset X-\eta_i. \tag{21}$$

By (19) and Lemma 4.3, the spaces S_i have bases of polynomial vectors of uniformly bounded degree in the rational coordinates of R. Pass to a subsequence on which their common dimension is r. Let V_0 be the affine k-space of coefficients of r such polynomial vectors, and let $t_i\in V_0(k)$ record a chosen basis of S_i.

The universal matrix over $(V_0)_K$ is obtained by evaluating the rational coordinates in those polynomial entries. Choose an $r\times r$ minor nonzero for infinitely many of the selected matrices, and retain that subsequence. On the open locus $U_0\subset(V_0)_K$ where this minor is nonzero, the rows define a rank-r subbundle. Pulling its projectivization back by $\Psi|_G$ gives a closed family

$$\mathcal{Z}\subset G\times U_0$$

whose fiber at t_i is Z_i.

Generic flatness gives a finite stratification of U_0 by reduced locally closed K-subschemes over which this family is flat; see [Stacks, Tag 0H3Y]. Retain a subsequence lying on one stratum W, and then a subsequence generic in an integral closed k-subvariety $V \subset V_0$. Constant points dense in V remain dense in V_K: expanding the coefficients of an equation in a finite k-linearly independent set reduces its vanishing to equations over k. Give $\overline{W}$ its reduced structure. Then $V_K \subset \overline{W}$ as schemes, since V_K is reduced, and

$$U = V_K \cap W$$

is open in V_K and contains all the retained t_i. The family $\mathcal{Z} \to U$ is flat and finitely presented. The open set U need not descend to k.

Define

$$\mathcal{R}_U = \{(\eta, t) \in G \times U : \eta + \mathrm{Supp}(\mathcal{Z}_t) \subset X\}.$$

Containment here is on supports of geometric fibers. This is a closed subset. Its complement is the image of the open subset of $G \times \mathcal{Z}$ where $\eta + z \notin X$. The projection to $G \times U$ is flat and finitely presented, hence open by [Stacks, Tag 01UA]. Give $\mathcal{R}_U$ its reduced structure and let $\mathcal{R}$ be its closure in $G \times V_K$. Over U this closure is still $\mathcal{R}_U$. In particular,

$$(\eta, t_i) \in \mathcal{R} \quad \Longleftrightarrow \quad \eta + Z_i \subset X. \tag{22}$$

The assertions follow from (21) and (22). All subsequences used here preserve genericity if the original x_i were generic in X. □

4.5. **Simultaneous evaluation over the constants.** Retain $\mathcal{R} \subset G \times V_K$ from Proposition 4.4. To use its constant parameters, we need a family over an enlarged constant field that records exactly the constant points of every fiber $\mathcal{R}_\eta$, for $\eta \in G(K)$. The construction below uses finitely many independent copies of K. The same copies work for all η because the parameter-coordinate functions in equations of $\mathcal{R}$ span a fixed finite-dimensional space.

Choose finite affine covers $V_a = \mathrm{Spec}\, E_a$ of V and $G_b = \mathrm{Spec}\, B_b$ of G. On $G_b \times (V_a)_K$, choose generators of the ideal of $\mathcal{R}$ and write them as

$$g_{abv} = \sum_\ell h_{abv\ell} \otimes e_{abv\ell} \in B_b \otimes_k E_a.$$

For each a, let $W_a \subset E_a$ be the k-span of all the finitely many elements $e_{abv\ell}$. Every specialized fiber equation belongs to $K \otimes_k W_a$. Choose an integer

$$N \geq \max\bigl(1, \max_a \dim_k W_a\bigr).$$

Since K/k is regular, the tensor product of N copies of K over k is a domain. Set

$$k' = \overline{\mathrm{Frac}(K \otimes_k \cdots \otimes_k K)}.$$

The tensor factors are abstract copies of K. Form the independent compositum $K' = Kk'$ and its algebraic closure $\overline{K'}$ as at the start of this section. A generic sequence of old points remains generic after this extension. Indeed, expand the finitely many coefficients of an equation over $\overline{K'}$ in an $\overline{K}$-linearly independent set. If the equation vanishes at infinitely many old points, every coefficient equation over $\overline{K}$ does also.

Let $\sigma_j : K \hookrightarrow k'$ be the embedding from the j-th tensor factor. Define

$$G_j = G \otimes_{K,\sigma_j} k', \qquad P = \prod_{j=1}^{N} G_j, \qquad \mathcal{R}_j = \mathcal{R} \otimes_{K,\sigma_j} k'.$$

The embeddings induce a homomorphism of groups

$$u : G(K) \longrightarrow P(k'), \qquad u(\eta) = (\sigma_1(\eta), \dots, \sigma_N(\eta)).$$

Define the closed subset

$$\mathcal{C} = \{(u_1, \dots, u_N, t) \in P \times V_{k'} : (u_j, t) \in \mathcal{R}_j \text{ for all } j\}.$$

For $\eta \in G(K)$, let $V_\eta \subset V$ be the reduced Zariski closure of $\mathcal{R}_\eta(\overline{K}) \cap V(k)$.

Proposition 4.5. *For every $\eta \in G(K)$,*

$$\mathcal{C}_{u(\eta),\mathrm{red}} = (V_\eta)_{k'}. \tag{23}$$

Proof. Fix η and a group chart containing it; its conjugate charts contain every $\sigma_j(\eta)$. On V_a, write a chosen equation of the fiber in the form

$$g = \sum_{\ell=1}^{s} b_\ell v_\ell, \qquad b_1, \dots, b_s \in K \text{ linearly independent over } k, \qquad v_\ell \in W_a, \quad s \le N.$$

At a k-point, g vanishes if and only if every v_ℓ vanishes. Since k is algebraically closed, the equations $v_\ell = 0$, for all chosen g, cut out V_η after taking the reduced structure. They also give

$$(V_\eta)_{\overline{K}} \subset (\mathcal{R}_\eta)_{\overline{K}}.$$

The matrix $(\sigma_j(b_\ell))_{j,\ell}$ has rank s. Its first s rows have determinant equal to the image of

$$\sum_{\pi \in \mathfrak{S}_s} \operatorname{sgn}(\pi)\, b_{\pi(1)} \otimes \cdots \otimes b_{\pi(s)} \in K^{\otimes_k s}.$$

Extend $b_1, \dots, b_s$ to a k-basis of K. The displayed summands are distinct basis tensors, so their sum is nonzero in every characteristic. It stays nonzero in k'. Hence the equations $\sigma_j(g) = 0$, for $1 \le j \le N$, are equivalent to the coefficient equations $v_\ell = 0$. Apply this argument to the chosen generators on every affine chart. On overlaps these coefficient equations give the same reduced closed set, namely the closure of the constant solutions. Moreover, V_η is geometrically reduced because k is algebraically closed. Taking reductions therefore gives the global equality (23). □

Finally, apply generic-flatness stratification to $\mathcal{C} \to P$ and retain a subsequence such that $u_i = u(\eta_i)$ lies in one reduced locally closed stratum U. On $U_{\overline{K'}} \times G_{\overline{K'}}$, put

$$I_U = \{(u, v) : \{v\} \times \mathcal{C}_{u,\mathrm{red}} \subset \mathcal{R}_{\overline{K'}}\}. \tag{24}$$

As in the proof of Proposition 4.4, this subset is closed: failure of containment is open in $\mathcal{C}_U \times G$, and its image under the flat projection is open. This argument uses the support of each fiber and does not require that the fibers be reduced or proper. The inclusion $(V_{\eta_i})_{\overline{K}} \subset (\mathcal{R}_{\eta_i})_{\overline{K}}$ and (20) now give

$$(u_i, \eta_i) \in I_U, \qquad t_i \in \mathcal{C}_{u_i}.$$

4.6. **Proof of bounded replacement.** The density hypothesis supplies a generic approximating sequence over any algebraically closed field. Fix a projective closure of X. For each integer $m \geq 1$, let E_m be the finite-dimensional space of degree-m homogeneous forms restricted to this closure. Evaluation at a point determines a line in E_m^*. At stage i, choose a point with $d_\Gamma(x_i) < 1/i$ whose evaluation, for every $m \leq i$, lies outside the spans of all sets of at most $\dim E_m - 1$ earlier evaluations. Each span is proper, and evaluations on X span E_m^*, so these are finitely many proper closed conditions. Density permits the choice. If a nonzero form vanished at infinitely many selected points, choose finitely many of their evaluations spanning all of them. Every later such evaluation would violate the construction. Hence the resulting sequence is generic.

Proof of Proposition 4.2. Choose a generic sequence (x_i) in $X(\overline{K})$ and points $\gamma_i \in \Gamma$ such that

$$\rho(x_i - \gamma_i) \longrightarrow 0.$$

Round the rational coefficients of γ_i as in §4.1. We obtain $\eta_i \in \Gamma_0 \subset G(K)$ such that the residual points $z_i = x_i - \eta_i$ satisfy

$$\sup_i \widehat{h}(z_i) < \infty, \qquad d_\Gamma(z_i) \longrightarrow 0.$$

Proposition 4.4 gives, after passage to a subsequence, an integral variety V/k, points $t_i \in V(k)$, and a closed subset $\mathcal{R} \subset G \times V_K$ with

$$(\eta_i, t_i) \in \mathcal{R}, \qquad z_i + \mathcal{R}_{t_i} \subset X. \tag{25}$$

Apply §4.5 to this family. It provides an independent extension of constants k'/k, a semiabelian variety P/k', and a group homomorphism

$$u : G(K) \longrightarrow P(k').$$

It also provides a closed family $\mathcal{C} \subset P \times V_{k'}$, a locally closed stratum $U \subset P$, and the relatively closed incidence $I_U \subset U_{\overline{K'}} \times G_{\overline{K'}}$ of (24). For the retained indices, writing $u_i = u(\eta_i)$, we have

$$(u_i, \eta_i) \in I_U, \qquad t_i \in \mathcal{C}_{u_i}.$$

The original sequence (x_i) remains generic over $\overline{K'}$, by the coefficient argument in §4.5.

Choose generators $P_1, \ldots, P_s$ of Γ_0, and let Λ be the subgroup of $(P_{\overline{K'}} \times G_{\overline{K'}})(\overline{K'})$ generated by

$$(u(P_j), P_j), \qquad 1 \leq j \leq s.$$

All tuples (u_i, η_i) belong to this finitely generated group. Pass to a subsequence which is generic in an integral closed subvariety $Z \subset P_{\overline{K'}} \times G_{\overline{K'}}$. Since I_U is closed relative to $U_{\overline{K'}} \times G_{\overline{K'}}$,

$$Z \cap (U_{\overline{K'}} \times G_{\overline{K'}}) \subset I_U. \tag{26}$$

To apply Proposition 3.3, we check its stabilizer hypothesis. Let D be the reduced identity component of

$$\mathrm{Stab}(Z) \cap (\{0\} \times G_{\overline{K'}}).$$

For $d \in D(\overline{K'})$, the point $(u_i, \eta_i + d)$ belongs to Z, and its first coordinate still belongs to U. By (26) and $t_i \in \mathcal{C}_{u_i}$, we have $(\eta_i + d, t_i) \in \mathcal{R}$. Equation (25) now gives

$$x_i + d = z_i + \eta_i + d \in X.$$

For fixed d, genericity implies $X + d \subset X$, hence $X + d = X$. Thus $D \subset \mathrm{Stab}(X)$. Since the latter is finite and D is connected and reduced, $D = 0$.

Proposition 3.3 gives

$$Z = \lambda + j(Y_{\overline{K'}}), \tag{27}$$

where $\lambda \in (\Lambda)^{\mathrm{div}}$, J/k' is semiabelian, $Y \subset J$ is closed and integral over k', and $j : J_{\overline{K'}} \to P_{\overline{K'}} \times G_{\overline{K'}}$ has finite kernel. Let $a : J \to P$ be the homomorphism over k' whose base change is $\mathrm{pr}_P \circ j$, and put $b = \mathrm{pr}_G \circ j$. Write $\lambda = (\lambda_P, \lambda_G)$.

The translating point has the required fields of definition:

$$\lambda_P \in P(k'), \qquad \lambda_G \in \Gamma \subset G(\overline{K}).$$

Indeed, some integer multiple of each coordinate belongs to the corresponding projection of Λ. Multiplication by this integer is finite. A finite fiber over an algebraically closed field acquires no new geometric points after extending that field. This applies also to inseparable multiplication. In particular, the full division hull of Γ_0 is unchanged when computed in $G(\overline{K'})$.

We now replace each center using a fiber defined over the new constant field. Set

$$E_i = \bigl(Y \cap a^{-1}(u_i - \lambda_P)\bigr)_{\mathrm{red}}.$$

This is a nonempty closed subvariety over k'. In fact, (27) gives $v_i \in E_i(\overline{K'})$ with $b(v_i) = \eta_i - \lambda_G$. For every $e \in E_i(k')$, the tuple $(u_i, \lambda_G + b(e))$ belongs to Z and has first coordinate in U. Equations (26) and (25) therefore give

$$y_{i,e} = \lambda_G + b(e) + z_i \in X(\overline{K'}).$$

Until the final descent, compute heights over K'/k', keeping Γ equal to the old division group. Since e is a constant point and b is a homomorphism from a constant semiabelian variety, $\rho(b(e)) = 0$. Consequently

$$\rho(y_{i,e}) \le \rho(\lambda_G) + \rho(z_i), \qquad d_\Gamma(y_{i,e}) = d_\Gamma(z_i) \longrightarrow 0. \tag{28}$$

The equality follows by translating by $\lambda_G \in \Gamma$ and then by the seminorm-zero point $b(e)$. By Lemma 4.1, the heights and distances of the old points z_i agree with their original values. Thus the convergence in (28) is uniform in e, and (1) gives a height bound independent of both i and e.

For each i, the constant points $E_i(k')$ are dense in $(E_i)_{\overline{K'}}$. Under the morphism $e \mapsto \lambda_G + b(e) + z_i$, their image therefore has closure containing the image of v_i, which is x_i. The union of the replacement sets over any tail of the indices has closure containing the corresponding tail of (x_i). It is consequently dense in $X_{\overline{K'}}$.

We have proved that there is $c_0 \ge 0$ such that, for every $\epsilon > 0$,

$$\{y \in X(\overline{K'}) : \widehat{h}_{K'/k'}(y) \le c_0,\ d_\Gamma(y) < \epsilon\}$$

is dense. This proves the proposition, with $R = c_0$. □

5. Descent of constant images

Use the independent extension k'/k, the fields $K' \subset \overline{K'}$, and the extended polarization from §4. A constant description after this extension must descend to the original constant field. The following argument applies to every such independent extension.

Retain the trace map $\mathrm{tr} : C_{\overline{K}} \to H \subset G$ from §2. The next lemma concerns the image of the trace, rather than a choice of trace object. This distinction permits finite inseparable kernels.

Lemma 5.1. *The following assertions hold.*

(1) The image of the Chow $\overline{K'}/k'$-trace of $G_{\overline{K'}}$ is $H_{\overline{K'}}$.

(2) Let $Z \subset G$ be a closed integral subvariety. Suppose that

$$Z_{\overline{K'}} = f(Y'_{\overline{K'}}),$$

where D is a semiabelian variety over k', $Y' \subset D$ is a closed integral subvariety over k', and $f : D_{\overline{K'}} \to G_{\overline{K'}}$ is a homomorphism with finite kernel. Then there is a closed integral subvariety $Y \subset C$, defined over k, such that

$$Z = \mathrm{tr}(Y_{\overline{K}}). \tag{29}$$

Proof. The old trace map extends to the new constant field, so $H_{\overline{K'}}$ is contained in the new trace image. To prove the opposite inclusion, suppose that there is a homomorphism

$$f : D_{\overline{K'}} \longrightarrow G_{\overline{K'}}$$

from a semiabelian variety D/k' whose image is not contained in $H_{\overline{K'}}$. We shall specialize its source to a semiabelian variety over k, while keeping G fixed.

Choose an integral variety V/k such that D spreads to a semiabelian scheme $\mathcal{D} \to V$. After enlarging the finitely generated field of definition inside k', we may assume that f is defined over a finite extension of $\overline{K}k(V)$. Normalize $V_{\overline{K}}$ in this extension and shrink to a nonempty open subscheme W on which f spreads to a homomorphism

$$\mathcal{D}_W \longrightarrow G \times_{\overline{K}} W.$$

Such a shrinking exists by finite presentation. The group identities also descend after shrinking, because they hold on the generic fiber.

Compose this homomorphism with $G \to G/H$. The inverse image of the complement of the identity section is a nonempty open subset of $\mathcal{D}_W$. Its image in W is open, since $\mathcal{D}_W \to W$ is smooth. The image of this open set in $V_{\overline{K}}$ contains a nonempty open subset. Moreover, $V(k)$ is dense in $V_{\overline{K}}$: an equation vanishing on these points can be expanded in finitely many coefficients linearly independent over k, and each resulting equation over k vanishes on $V(k)$.

We may therefore choose $v \in V(k)$ and an $\overline{K}$-point of W above v at which the specialized homomorphism has image outside H. This gives a homomorphism $(\mathcal{D}_v)_{\overline{K}} \to G$ from a constant semiabelian variety whose image is not contained in the trace image, contrary to the universal property of the trace. This proves the first assertion. No assertion about the trace of G/H is needed.

For the second assertion, the first gives $f(D_{\overline{K'}}) \subset H_{\overline{K'}}$, and hence $Z \subset H$. Choose an isogeny $j : H \to C_{\overline{K}}$ and an integer $m \geq 1$ such that

$$j \circ \mathrm{tr} = [m]_C.$$

The homomorphism $j \circ f$ is between semiabelian varieties defined over k'. It is therefore defined over k', by the full faithfulness of extension of scalars for semiabelian varieties [BBP16, Fact 2.4(iii) and §2.2.2]. Since $j \circ f$ has finite kernel, it is finite onto its image. Consequently $j(Z_{\overline{K'}})$ is a closed subvariety defined over k'.

Consider the reduced inverse image

$$E = (\mathrm{tr}^{-1}(Z))_{\mathrm{red}} \subset C_{\overline{K}}.$$

It satisfies

$$E_{\overline{K'}} \subset \left([m]^{-1}(j(Z_{\overline{K'}}))\right)_{\mathrm{red}}.$$

The right-hand side is defined over k'. Both sides have pure dimension $\dim Z$, since isogenies are finite and flat. Thus $E_{\overline{K'}}$ is a union of irreducible components of the right-hand side. These components are defined over the algebraically closed field k', so $E_{\overline{K'}}$ is defined over k'.

We now use that E is also defined over $\overline{K}$. Choose a projective compactification of C and an embedding defined over k. In each degree, the homogeneous ideal of the closure of E is a subspace defined over both $\overline{K}$ and k'. After dividing its Plücker coordinates by a nonzero coordinate, all their ratios belong to $\overline{K} \cap k' = k$. Each of these subspaces therefore descends to k. Finite generation of the ideal shows that E is the base change of a closed subvariety of C over k. Choose an irreducible component whose image under tr is Z. This gives Y and proves (29). □

6. Proof of the main theorem

Proof of Theorem 1.1. Suppose first that X has dense Γ-approximation. The quotient reduction in §2 reduces the assertion to the case of finite stabilizer. Proposition 4.2 gives an independent extension k'/k and a uniform height bound for dense approximation on $X_{\overline{K'}}$, using the original group Γ.

Proposition 2.7, applied over K'/k', gives

$$X_{\overline{K'}} = \gamma + f(W_{\overline{K'}}),$$

where $\gamma \in \Gamma$, W is a closed integral subvariety of a semiabelian variety over k', and f has finite kernel. The full division hull does not acquire new points after extension of the algebraically closed field $\overline{K}$, since multiplication by a positive integer is finite. Thus γ is an old point. Lemma 5.1, applied to $X - \gamma$, gives

$$X - \gamma = \mathrm{tr}(Y_{\overline{K}})$$

for a closed integral $Y \subset C = G^{\overline{K}/k}$ over k. This proves (2) in the finite-stabilizer case. Applying this conclusion to $Q = G_{\overline{K}}/B$ and $q(\Gamma)$, and using (4) to lift the translating point, proves it in general.

Conversely, suppose that (2) holds. The translate $X' = X - \gamma$ has connected reduced stabilizer B, and

$$q(X') = \mathrm{tr}_Q(Y_{\overline{K}}).$$

Since tr_Q has finite kernel, X' is special in the sense of semiabelian geometric Bogomolov, with zero torsion translator in the quotient. By [LY25, Theorem 1.1], X' has a dense set of points of arbitrarily small height. Translating these points by $\gamma \in \Gamma$ proves condition (1). □

6.1. **A consequence.** The theorem also gives a uniform threshold outside finitely many special subvarieties.

Corollary 6.1. *For every closed subvariety $V \subset G$, there exist $\epsilon_0 > 0$ and finitely many integral Γ-special subvarieties $V_j \subset V$ such that*

$$\{x \in V(\overline{K}) : d_\Gamma(x) < \epsilon_0\} \subset \bigcup_j V_j(\overline{K}).$$

If V is integral and not Γ-special, the subvarieties V_j can all be taken proper.

Proof. We argue by induction on dimension, treating the finitely many irreducible components separately. A Γ-special component can itself be included in the union. For an integral component which is not Γ-special, the theorem gives $\epsilon > 0$ such that the closure of its points satisfying $d_\Gamma(x) < \epsilon$ is proper. Apply the induction hypothesis to the finitely many irreducible components of this closure, and take the minimum of their thresholds and ϵ. In dimension zero, a point of distance zero is Γ-special by Lemma 2.3, and a point of positive distance is excluded by choosing a smaller threshold. The union is allowed to be empty. □

REFERENCES

[BBP16] Franck Benoist, Elisabeth Bouscaren, and Anand Pillay. Semiabelian varieties over separably closed fields, maximal divisible subgroups, and exact sequences. *Journal of the Institute of Mathematics of Jussieu*, 15(1):29–69, 2016.

[CGSZ21] Pietro Corvaja, Dragoş Ghioca, Thomas Scanlon, and Umberto Zannier. The dynamical Mordell–Lang conjecture for endomorphisms of semiabelian varieties defined over fields of positive characteristic. *Journal of the Institute of Mathematics of Jussieu*, 20(2):669–698, 2021.

[Con06] Brian Conrad. Chow's K/k-image and K/k-trace, and the Lang–Néron theorem. *L'Enseignement Mathématique*, 52(1–2):37–108, 2006.

[Hru96] Ehud Hrushovski. The Mordell–Lang conjecture for function fields. *Journal of the American Mathematical Society*, 9(3):667–690, 1996.

[Lau87] Michel Laurent. équations exponentielles polynômes et suites récurrentes linéaires. In *Journées arithmétiques de Besançon*, number 147–148 in Astérisque, pages 121–139. Société mathématique de France, 1987.

[Liu] Long Liu. Representability of smooth-site direct images and Chow trace of semi-abelian varieties. *In preparation.*

[Liu24] Long Liu. Chow trace of 1-motives and the Lang-Néron groups. *Preprint*, arXiv:2404.13200, 2024. Accepted by *Algebra & Number Theory*.

[Liu26] Long Liu. Mordell-Lang plus geometric Bogomolov. *Preprint*, arXiv:2609.38001, 2026.

[LY25] Wenbin Luo and Jiawei Yu. Geometric Bogomolov conjecture for semiabelian varieties, 2025. *Preprint*, arXiv:2505.07193, 2025.

[McQ95] Michael McQuillan. Division points on semi-abelian varieties. *Inventiones Mathematicae*, 120(1):143–159, 1995.

[Mor01] Atsushi Moriwaki. A generalization of conjectures of Bogomolov and Lang over finitely generated fields. *Duke Mathematical Journal*, 107(1):85–102, 2001.

[MS04] Rahim Moosa and Thomas Scanlon. F-structures and integral points on semiabelian varieties over finite fields. *American Journal of Mathematics*, 126:473–522, 2004.

[Poo99] Bjorn Poonen. Mordell–Lang plus Bogomolov. *Inventiones Mathematicae*, 137(2):413–425, 1999.

[Rém03] Gaël Rémond. Approximation diophantienne sur les variétés semi-abéliennes. *Annales scientifiques de l'École Normale Supérieure*, 36(2):191–212, 2003.

[Stacks] The Stacks Project authors. Stacks Project. https://stacks.math.columbia.edu/, 2026.

[XY22] Junyi Xie and Xinyi Yuan. Geometric Bogomolov conjecture in arbitrary characteristics. *Inventiones Mathematicae*, 229:607–637, 2022.

INSTITUTE FOR THEORETICAL SCIENCES, WESTLAKE UNIVERSITY, 600 DUNYU ROAD, XIHU DISTRICT, HANGZHOU 310030, ZHEJIANG PROVINCE, P.R. CHINA.

Email address: liulong@westlake.edu.cn