CHOW TRACE OF 1-MOTIVES AND THE LANG-NÉRON GROUPS

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With an appendix by Bruno Kahn

ABSTRACT. We show that in the case of primary field extensions, the extension of scalars of Deligne 1-motives admits a left adjoint, called Chow image, and a right adjoint, called Chow trace. This generalizes Chow's results on abelian varieties. Then we study the Chow trace in the framework of Voevodsky's triangulated categories of (étale) motives. With respect to the 1-motivic t-structure on the category of Voevodsky's homological 1-motives, the zero-th direct image of an abelian variety is given by the Chow trace, and the first direct image is the 0-motive defined by the (geometric) Lang-Néron group.

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1. Introduction

1.1. **Background.** Let K/k be a field extension, K_s be a separable closure of K and k_s be the separable closure of k in K_s. For a finite separable field extension K/k, the field k_s is K_s itself and the absolute Galois group $\mathrm{Gal}(K_s/K)$ is canonically an open subgroup of $\mathrm{Gal}(k_s/k)$. Then the extension of scalars of discrete Galois modules admits a left adjoint and a right adjoint, both of which are given by the induced modules in the sense of [Ser02, Chapter I, 2.5]. A deeper result is the existence of a left adjoint and a right adjoint to the extension of scalars of abelian varieties, both of which are given by the Weil restriction. See, for example, [Kah18, Th. 4.2 and 4.3].

Another interesting case is when K/k is a primary extension of fields, which means that the algebraic closure of k in K is purely inseparable over k. Then the canonical homomorphism of absolute Galois groups $\mathrm{Gal}(K_s/K) \to \mathrm{Gal}(k_s/k)$ is surjective. The extension of scalars of discrete Galois modules admits a left adjoint and a right adjoint, given by $\mathrm{Gal}(K_s/Kk_s)$ coinvariants and $\mathrm{Gal}(K_s/Kk_s)$ invariants respectively. A deeper result ([Cho55]) is the existence of a left adjoint and a right adjoint to the extension of scalars of abelian varieties, called Chow image and Chow trace respectively. Lang and Néron ([LN59]) proved a relative version of Mordell-Weil theorem using Chow's trace: Let K/k be a finitely generated regular field extension. Let A be an abelian variety over K and $\pi_* A$ be its K/k-trace. Then the Lang-Néron group

$$\mathrm{LN}(A, K/k) := A(K)/(\pi_* A)(k)$$

is a finitely generated abelian group. See [Con06] and [Kah06, Appendices A and B] for 'modern' proofs using Grothendieck's theory of schemes and fpqc descent.

1.2. **Main results.** The existence of Chow's trace can be generalized to Deligne 1-motives. Recall [Del74, 10.1.10] that a Deligne 1-motive over k is a two-term complex of group schemes $[L \to G]$, where L is a lattice and G is a semi-abelian variety. Here, a lattice means a commutative étale group scheme L over k such that $L(k_s)$ is a finitely generated free $\mathbb{Z}$-module, and a semi-abelian variety is a commutative algebraic group which is an extension of an abelian variety by a torus. A morphism of Deligne 1-motives is defined to be a commutative square in the obvious sense. Denote the category of Deligne 1-motives by $\mathsf{M}_1(k)$. The base change of group schemes induces a base change functor of Deligne 1-motives. The following result answers positively the expectation from [Kah18, bottom of p. 82].

Theorem 1.2.1 (Theorems 2.3.8 and 2.4.2)**.** *Let K be a primary field extension of k. Then*

(1) *the extension of scalars of Deligne 1-motives*

$$\begin{aligned} \pi^* : \mathsf{M}_1(k) &\longrightarrow \mathsf{M}_1(K) \\ [L \to G] &\longmapsto [L_K \to G_K] \end{aligned}$$

is fully faithful;

(2) *π^* has a left adjoint $\pi_\sharp^{\mathsf{M}_1}$ and a right adjoint $\pi_*^{\mathsf{M}_1}$.*

Our functors $\pi_\sharp^{\mathsf{M}_1}$ and $\pi_*^{\mathsf{M}_1}$ recover some classical constructions (Corollaries 2.4.4 and 2.4.6, and Proposition 2.4.8), such as Chow's image and trace of abelian varieties. Thus we call these two functors Chow image and Chow trace respectively. A key ingredient of the existence of Chow image and Chow trace is the fact that primary field extensions will not bring new semi-abelian subvarieties (Theorem 2.3.12).

We also want to study the derived functors of π_*. However, the category of Deligne 1-motives is neither abelian nor big enough. Let Λ be $\mathbb{Z}[1/p]$, the localization of $\mathbb{Z}$ by inverting p the exponential characteristic of k. Thanks to the work of Voevodsky, Orgogozo, Barbieri-Viale, Kahn and Ayoub ([Voe00], [Org04], [BVK16], [Ayo11]), $\mathsf{M}_1(k)\otimes_{\mathbb{Z}}\Lambda$ is a full subcategory of the heart of a 1-motivic t-structure on Voevodsky's triangulated category $\mathsf{DM}_{\le 1}(k,\Lambda)$ of étale homological 1-motives, i.e., the localizing subcategory of $\mathsf{DM}^{\mathrm{eff}}_{\text{ét}}(k,\Lambda)$ generated by the motives $M(X)$ for $\dim X \le 1$.

By the work of Ayoub and Barbieri-Viale [ABV09], $\mathsf{DM}_{\le 1}(k,\Lambda)$ is canonically equivalent to the unbounded derived category of (étale) 1-motivic sheaves $\mathsf{HI}_{\le 1}(k,\Lambda)$, which is the smallest co-complete Serre subcategory of the category of étale sheaves with transfers containing lattices and étale sheaves represented by semi-abelian varieties. And every smooth curve C defines a 1-motivic sheaf $h_0^{\text{ét}}(C)$, which will form a system of generators of $\mathsf{HI}_{\le 1}(k,\Lambda)$. The category of 1-motivic sheaves contains the category $\mathsf{HI}_{\le 0}(k,\Lambda)$ of 0-motivic sheaves, which is equivalent to the category of sheaves of Λ-modules on the site $(\operatorname{Spec} k)_{\text{ét}}$. Ayoub and Barbieri-Viale showed that the inclusion $\delta\colon \mathsf{HI}_{\le 0} \hookrightarrow \mathsf{HI}_{\le 1}$ admits a left adjoint π_0, which is constructed using the scheme of connected components. Then we will have an analogue of the connected-étale exact sequence

$$0 \to \mathscr{F}^0 \to \mathscr{F} \to \pi_0(\mathscr{F}) \to 0.$$

Let K/k be a field extension. Then the inverse image functor

$$e^*\colon \mathsf{HI}_{\le 1}(k,\Lambda) \longrightarrow \mathsf{HI}_{\le 1}(K,\Lambda)$$
$$h_0^{\text{ét}}(C) \longmapsto h_0^{\text{ét}}(C_K)$$

admits a right adjoint e_*, which has a total right derived functor Re_*. Similarly, we also have a direct image functor ε_* for 0-motivic sheaves.

Theorem 1.2.2 (Theorem 3.6.13)**.** *For $\mathscr{F} \in \mathsf{HI}_{\le 0}(K,\Lambda)$, we have a canonical isomorphism*

$$\delta R^i \varepsilon_* \mathscr{F} \xrightarrow{\sim} R^i e_* \delta \mathscr{F}.$$

In particular, if K/k is primary, then $R^i e_* \delta\mathscr{F}$ is the 0-motivic sheaf associated with the $\mathrm{Gal}(k_s/k)$-module $H^i(\Gamma, \mathscr{F}_{K_s})$, where $\Gamma = \mathrm{Gal}(K_s/Kk_s)$.

The key point of the proof is a smooth base change theorem for non-torsion étale sheaves (Corollary A.2.5), whose proof will be given in Appendix A. Besides, we shall need some knowledge about model categories to study the unbounded derived functors used in the proof.

Theorem 1.2.3 (Theorems 3.8.7 and 3.8.10)**.** *Let K/k be a field extension.*

(1) *If A is an abelian variety over K, then $R^i e_* A$ is a torsion 0-motivic sheaf for $i \ge 1$.*

(2) *If K/k is a primary field extension, then the connected-étale exact sequence associated with e_*A is*
$$0 \to \pi_* A \to e_* A \to \mathrm{LN}(A, Kk_s/k_s)_\Lambda \to 0.$$

To prove the first assertion, we shall use the fact that $H^i_{\text{ét}}(X, \mathscr{F})$ is torsion for $i > \dim X$, Raynaud's theorem that $H^1_{\text{ét}}(X, A)$ is torsion for X noetherian regular and A an abelian scheme over X, and Suslin's rigidity theorem [MVW06, Theorem 7.20]. For the second assertion, we shall check that Chow trace is the connected component of the direct image by using a structure theorem of 1-motivic sheaves, which is due to Ayoub, Barbieri-Viale and Kahn, and using the universal property of Chow trace. The Lang-Néron theorem now can be used to deduce the finiteness of e_*A when K/k is a finitely generated regular extension (see Corollary 3.8.8).

We shall refine the 1-motivic t-structure with $\mathbb{Q}$-coefficients in [Ayo11] to $\mathbb{Z}[1/p]$-integral coefficients. An object in $D(\mathsf{HI}_{\le 1}(k,\Lambda))$ is in the heart of the 1-motivic t-structure if and only if it is quasi-isomorphic to a two-term complex $[L \to G]$ concentrated in degrees $0,1$ with $\ker(L \to G)$ a 0-motivic sheaf and $\mathrm{coker}(L \to G)$ a connected 1-motivic sheaf. We will call it a 0-motive if it is quasi-isomorphic to $[L \to 0]$ with L a 0-motivic sheaf. Using a proposition (4.2.9) comparing the two t-structures on $D(\mathsf{HI}_{\le 1})$, we can translate the above theorems to some results on the higher direct images relative to the 1-motivic t-structure.

Denote by $R^i e_*$ (resp. ${}^m R^i e_* = [L^i \to G^i]$) the cohomology of Re_* relative to the standard (resp. 1-motivic) t-structure on $D(\mathsf{HI}_{\le 1}(k,\Lambda))$.

Theorem 1.2.4 (Theorem 4.4.1)**.** *Let K/k be a field extension and let L be a 0-motivic sheaf over K. Then*
$${}^m R^i e_*[L \to 0] = [R^i e_* L \to 0].$$
In particular, ${}^m R^i e_*[L \to 0]$ is a torsion 0-motive for $i \ge 1$.

Theorem 1.2.5 (Theorem 4.4.6)**.** *Let K/k be a primary field extension and let A be an abelian variety over K. Then*
$${}^m R^i e_*[0 \to A] = \begin{cases} [0 \to \pi_* A], & \text{if } i = 0; \\ [\mathrm{LN}(A, Kk_s/k_s) \to 0], & \text{if } i = 1; \\ [R^{i-1} e_* A \to 0], & \text{if } i \ge 2. \end{cases}$$
In particular, ${}^m R^0 e_*[0 \to A]$ is a constructible 1-motive, and ${}^m R^i e_*[0 \to A]$ are torsion 0-motives for $i \ge 2$. Moreover, if K/k is a finitely generated regular extension, then ${}^m R^1 e_*[0 \to A]$ is a constructible 0-motive.

Theorem 1.2.6 (Theorem 4.4.7)**.** *Let X be a smooth projective and geometrically connected variety over k and let K be the function field of X. Then*
$${}^m R^i e_*[0 \to \mathbb{G}_m] = \begin{cases} [0 \to \mathbb{G}_m], & \text{if } i = 0; \\ 0, & \text{if } i = 2; \\ [R^{i-1} e_* \mathbb{G}_m \to 0], & \text{if } i \ge 3. \end{cases}$$
Moreover, with $\mathbb{Q}$-coefficients, we have
$${}^m R^1 e_*[0 \to \mathbb{G}_m] = [\mathrm{Div}^0(X_{k_s}) \to \mathrm{Pic}^0_{X/k}].$$

The following result can be viewed as a generalization of the Lang-Néron theorem for certain Deligne 1-motives.

Theorem 1.2.7 (Theorem 4.4.10)**.** *Let K/k be a finitely generated regular field extension and let $M = [L \to A]$ be a Deligne 1-motive over K where A is an abelian variety. Write $\Gamma = \mathrm{Gal}(K_s/Kk_s)$. Then we have an exact sequence of 0-motivic sheaves*

$$0 \to X \to \pi_0(R^1 e_* M) \to Y \to 0,$$

where

$$X = \mathrm{coker}(L(K_s)^\Gamma \to \mathrm{LN}(A, Kk_s/k_s))$$

and

$$Y = \ker(H^1(\Gamma, L(K_s)) \to R^1 e_* A).$$

In particular,

(1) *$\pi_0(R^1 e_* M)(k_s)$ is a finitely generated $\mathrm{Gal}(k_s/k)$-module;*
(2) *${}^m R^1 e_* M = [\pi_0(R^1 e_* M) \to 0]$ is a constructible 0-motive.*

This fails in the presence of tori; see Theorem 1.2.6. One can recover a finite generation statement if one replaces the function field by a smooth model.

The story in the case when K/k is a finite extension can be found in [PL19, A.17 and Lemma 2.22]. In fact, he studied it in the more general setting when $f\colon X \to Y$ is a finite étale morphism.

1.3. **Conventions and notations.** We shall use the following categorical notions in the spirit of [KS06, Definition 8.3.21].

Definition 1.3.1. Let $\mathcal{A}$ be an abelian category and $\mathcal{B}$ be a full subcategory of $\mathcal{A}$.

(1) We say that $\mathcal{B}$ is a Serre subcategory of $\mathcal{A}$ if it is closed under subobjects, quotients and extensions.
(2) We say that $\mathcal{B}$ is a thick subcategory of $\mathcal{A}$ if it is closed under kernels, cokernels and extensions.
(3) We say that $\mathcal{B}$ is a fully abelian subcategory of $\mathcal{A}$ if it is additive and closed under kernels and cokernels, equivalently, $\mathcal{B}$ is an abelian category and the embedding functor is exact.

Remark 1.3.2. (1) Clearly, a Serre subcategory is thick, and a thick subcategory is a fully abelian subcategory.

(2) In [Stacks, Definition 02MO], a (strictly full) thick subcategory is called a weak Serre subcategory.

We list some notations used in this chapter:

- k, field of exponential characteristic p
- Λ, the ring $\mathbb{Z}[1/p]$
- $\mathsf{M}_0(k)$ (resp. ${}^t\mathsf{M}_0(k)$), category of lattices (resp. of constructible group schemes) (2.1.1)
- $\mathsf{Tori}(k)$, category of tori
- $\mathsf{AV}(k)$ (resp. $\mathsf{SAV}(k)$), category of abelian varieties (resp. of semi-abelian varieties)
- ${}_nG$, kernel of the multiplication by n on a semi-abelian variety
- $\mathsf{M}_1(k)$, category of Deligne 1-motives
- Sm/k, category of smooth schemes separated of finite type over k
- $(\mathsf{Sm}/k)_{\le n}$, full subcategory of Sm/k consisting of schemes with dimension $\le n$

- $\mathsf{Shv}_{\text{ét}}(\mathcal{C},\Lambda)$, category of sheaves of Λ-modules on the étale site $\mathcal{C}_{\text{ét}}$
- $\mathsf{Shv}^{\text{tr}}_{\text{ét}}(k,\Lambda)$, category of étale sheaves with transfers on Sm/k (reviewed in §3.2)
- $\mathsf{Shv}^{\text{tr}}_{\text{ét}}(k_{\leq n},\Lambda)$, category of étale sheaves with transfers on $(\mathsf{Sm}/k)_{\leq n}$ (reviewed in §3.2)
- $\mathsf{HI}_{\text{ét}}(k,\Lambda)$, category of homotopy invariant sheaves (reviewed in §3.3)
- $\mathsf{HI}_{\leq n}(k,\Lambda)$, category of n-motivic sheaves (reviewed in §3.4)
- $\mathsf{DM}^{\text{eff}}_{\text{ét}}(k,\Lambda)$, Voevodsky category of effective étale motives
- $\mathsf{DM}_{\leq n}(k,\Lambda)$, the localizing subcategory of $\mathsf{DM}^{\text{eff}}_{\text{ét}}(k,\Lambda)$ generated by the motives $M(X)$ for $\dim X \leq n$
- $\pi^*\colon \mathsf{M}_1(k) \to \mathsf{M}_1(K)$, extension of scalars of Deligne 1-motives induced by base change of schemes
- π_*, right adjoint to π^*, i.e., Chow trace
- $\pi_\sharp$, left adjoint to π^*, i.e., Chow image
- $\gamma_*\colon \mathsf{Shv}^{\text{tr}}_{\text{ét}}(k,\Lambda) \to \mathsf{Shv}_{\text{ét}}(\mathsf{Sm}/k,\Lambda)$, the forgetful functor
- γ^*, left adjoint to γ_*
- $\iota\colon \mathsf{HI}^{\text{tr}}_{\text{ét}}(k,\Lambda) \to \mathsf{Shv}^{\text{tr}}_{\text{ét}}(k,\Lambda)$, the inclusion functor
- $h_0^{\text{ét}}\colon \mathsf{Shv}^{\text{tr}}_{\text{ét}}(k,\Lambda) \to \mathsf{HI}^{\text{tr}}_{\text{ét}}(k,\Lambda)$, a left adjoint to ι
- $\iota_n\colon \mathsf{HI}_{\leq n}(k,\Lambda) \to \mathsf{HI}^{\text{tr}}_{\text{ét}}(k,\Lambda)$, the inclusion functor
- $\sigma_n^*\colon \mathsf{Shv}^{\text{tr}}_{\text{ét}}(k_{\leq n},\Lambda) \leftrightarrows \mathsf{Shv}^{\text{tr}}_{\text{ét}}(k,\Lambda)\colon \sigma_{n*}$, extension and restriction functors
- e^*_{tr}, e^{tr}_*, inverse image and direct image functors of sheaves with transfers on Sm
- $e^*_{\leq n}$, $e^{\leq n}_*$, inverse image and direct image functors of sheaves (with transfers) on $(\mathsf{Sm})_{\leq n}$
- e^*_{HI}, e^{HI}_*, inverse image and direct image functors of homotopy invariant sheaves
- e^*_n, e_{n*}, inverse image and direct image functors of n-motivic sheaves
- $\pi_0\colon \mathsf{Shv}^{\text{tr}}_{\text{ét}}(k,\Lambda) \to \mathsf{HI}_{\leq 0}(k,\Lambda)$, left adjoint to the inclusion functor

2. Chow image and Chow trace of Deligne 1-motives

In this section, we study the extension of scalars of Deligne 1-motives. More precisely, we will show some full faithfulness results and will construct the Chow image and Chow trace of Deligne 1-motives.

2.1. **Commutative étale group schemes.** In this subsection, we fix our notations and recall some well-known and not so well-known facts on étale group schemes over a field.

Let k be a field and let k_s be a separable closure of k.

Definition 2.1.1. Let L be a commutative étale group scheme over k.

(1) We say that L is a constructible group scheme if $L(k_s)$ is a finitely generated abelian group.
(2) We say that L is a lattice if $L(k_s)$ is a finitely generated free abelian group.
(3) Denote the category of constructible group schemes (resp. lattices) over k by ${}^t\mathsf{M}_0(k)$ (resp. $\mathsf{M}_0(k)$), where morphisms are homomorphisms between group schemes.

Remark 2.1.2. Constructible group schemes are called discrete group schemes in [BVK16]. We call such group schemes constructible to emphasize the finiteness and to avoid the confusion with discrete Galois modules.

Remark 2.1.3. By [DG70, Chapitre II, §5, Proposition 1.4], a group scheme G locally of finite type over k is étale if and only if $G^0 \simeq \operatorname{Spec} k$. Thus if G is étale, then every subgroup scheme is also étale. Moreover, if G is a constructible group scheme (resp. a lattice, resp. a finite étale group scheme), then so is every subgroup scheme.

The following result gives a concrete way to study commutative étale group schemes over k:

Proposition 2.1.4 ([DG70, Chapitre II, §5, Proposition 1.7])**.** *The functor $L \mapsto L(k_s)$ is an equivalence from the category of étale group schemes to the category of discrete* $\mathrm{Gal}(k_s/k)$*-groups. Moreover, via this functor, constructible group schemes (resp. lattices, resp. commutative finite étale group schemes) correspond to finitely generated (resp. finitely generated free, resp. finite) abelian groups with continuous* $\mathrm{Gal}(k_s/k)$*-actions.*

Let S be a scheme. For a commutative group scheme G over S, we have a group functor

$$\begin{aligned} G^\vee\colon \mathsf{Sch}/S &\longrightarrow \{\text{abelian groups}\} \\ X &\longmapsto \mathrm{Hom}_X(G_X, \mathbb{G}_{m,X}), \end{aligned}$$

called Cartier duality of G.

Recall that a group scheme over k is called a torus if $T_{\bar{k}} \simeq \mathbb{G}^n_{m,\bar{k}}$ for some $n \in \mathbb{N}$. Tori are commutative, connected, affine, smooth and of finite type over k. Denote by $\mathsf{Tori}(k)$ the category of tori over k. We have the following duality theorem:

Theorem 2.1.5. *The functors $T \mapsto T^\vee$ and $L \mapsto L^\vee$ are anti-equivalences, quasi-inverses of each other, between* $\mathsf{Tori}(k)$ *and* $\mathsf{M}_0(k)$.

Proof. By [DG70, Chapitre IV, §1, Corollaire 3.3], these two functors are anti-equivalences, quasi-inverses of each other, between the category of the group schemes of multiplicative type over k and the category of commutative étale group schemes over k. Then by [DG70, Chapitre IV, §1, Corollaire 3.9 (a)], tori correspond to lattices. □

2.2. **Semi-abelian varieties.** In this subsection, we shall work in the abelian category of commutative algebraic groups (i.e., group schemes of finite type) over a field k; see [SGA 3_{I}, Exposé $\mathrm{VI_A}$, Théorème 5.4.2] for a proof that this category is abelian. By [DG70, Chapitre II, §5, Théorème 2.1], an algebraic group over k is smooth if and only if it is geometrically reduced.

Lemma 2.2.1. *Let T be a smooth connected affine algebraic group over k and let A be an abelian variety over k. Then there is neither a nontrivial homomorphism from T to A nor a nontrivial homomorphism from A to T.*

Proof. For any homomorphism $f\colon T \to A$, the quotient $T/\ker(f)$ inherits the properties of being smooth connected and affine from T by [SGA 3_{I}, Exposé $\mathrm{VI_B}$, Proposition 9.2(xii) and Théorème 11.17]. Since A is proper, its closed subgroup $T/\ker(f)$ is also proper. Because $T/\ker(f)$ is both proper and affine, it is finite. Since $T/\ker(f)$ is finite étale and connected, it is isomorphic to $\operatorname{Spec} k$, which means that f is trivial.

Similarly, for any homomorphism $g\colon A \to T$, the quotient $A/\ker(g)$ inherits the properties of being smooth connected and proper from A. Since T is affine, its closed subgroup $A/\ker(g)$ is also affine. Because $A/\ker(g)$ is both proper and affine, it is finite. Since $A/\ker(g)$ is finite étale and connected, it is isomorphic to $\operatorname{Spec} k$, which means that g is trivial. □

Remark 2.2.2. When T is a torus, we can also use [Mil86, Corollary 3.9] to see that there exists no nontrivial (homo)morphism from T to A.

Let G be a smooth connected commutative algebraic group over k. By Chevalley's theorem (see, e.g., [Con02]), $G_{\overline{k}}$ is uniquely an extension of an abelian variety by a smooth connected affine group $G^{\mathrm{aff}}_{\overline{k}}$. By Lemma 2.2.1, these are functorial in G.

Definition 2.2.3. A commutative algebraic group G over k is called a semi-abelian variety if it can be represented by an extension

$$0 \to T \to G \to A \to 0,$$

where T is a torus and A is an abelian variety. Since T and A are both smooth and connected, so is G.

Remark 2.2.4. (1) We call T and A the toric part and abelian part of G respectively. By Lemma 2.2.1, these are functorial in G. In particular, the groups T and A are uniquely determined by G.

(2) By [BLR90, bottom of p. 178], a smooth connected commutative algebraic group G is semi-abelian if and only if $G^{\mathrm{aff}}_{\overline{k}}$ is a torus.

Lemma 2.2.5. (1) *The quotients and smooth connected subgroups of a torus are tori.*

(2) *The quotients and smooth connected subgroups of an abelian variety are abelian varieties.*

(3) *The quotients and the smooth connected subgroups of a semi-abelian variety are semi-abelian varieties.*

Proof. (1) This assertion follows from [DG70, Chapitre IV, §1, Corollaires 2.4 and 3.9 (a)].

(2) The closed subgroups of abelian varieties are proper over k. If they are smooth connected, then they are abelian varieties by definition. The quotients of abelian varieties inherit the properties of being smooth connected and proper over k (see [SGA 3_{I}, Exposé $\mathrm{VI_B}$, Proposition 9.2(xii)]). Thus they are abelian varieties.

(3) By Remark 2.2.4 (2), we may and do assume that k is algebraically closed. Let G be a semi-abelian variety over k, i.e., G^{aff} is a torus.

If G' is a smooth connected subgroup of G, then G'^{aff} is a closed subgroup of the torus G^{aff}. Thus G'^{aff} is also a torus, which implies that G' is a semi-abelian variety.

Let $f\colon G \to G''$ be a surjection. Let H be the categorical image of the induced morphism $G^{\mathrm{aff}} \to G''^{\mathrm{aff}}$ in the abelian category of commutative algebraic groups over k. Since G/G^{aff} is an abelian variety, its quotient G''/H is also an abelian variety. Then the smooth connected subgroup G''^{aff}/H of G''/H is an abelian variety. But G''^{aff}/H inherits the property of being affine from G''^{aff}. Thus G''^{aff}/H is trivial, which implies that the morphism $G^{\mathrm{aff}} \to G''^{\mathrm{aff}}$ is an epimorphism. Then G''^{aff} inherits the property of being a torus from G^{aff}. Thus G'' is a semi-abelian variety. ☐

Lemma 2.2.6. *Let k be a field and G be a semi-abelian variety over k. Let n be a positive integer prime to* $\mathrm{char}(k)$ *and let ${}_nG$ be the kernel of multiplication by n on G. Then ${}_nG$ is finite étale over k.*

Proof. See [BLR90, §7.3, Lemmas 1 and 2]. ☐

Lemma 2.2.7. *Let k be an algebraically closed field and let G be a semi-abelian variety over k. Let n be a positive integer prime to* $\mathrm{char}(k)$*. Then*

$$\#_n G(k) = n^{t+2a},$$

where t (resp. a) is the dimension of the toric part (resp. abelian part) of G.

Proof. Consider the following commutative diagram with exact rows in the abelian category of commutative algebraic groups

$$\begin{tikzcd} 0 \arrow[r] & T \arrow[r] \arrow[d, "n"'] & G \arrow[r] \arrow[d, "n"'] & A \arrow[r] \arrow[d, "n"'] & 0 \\ 0 \arrow[r] & T \arrow[r] & G \arrow[r] & A \arrow[r] & 0. \end{tikzcd}$$

Using the snake lemma and noting that $n\colon T \to T$ is an epimorphism, we have the following exact sequence

$$0 \to {}_nT \to {}_nG \to {}_nA \to 0.$$

By Lemma 2.2.6, the group schemes ${}_nT$, ${}_nG$ and ${}_nA$ are finite étale over k. Using Proposition 2.1.4, we obtain the following exact sequence of finite abelian groups

$$0 \to {}_nT(k) \to {}_nG(k) \to {}_nA(k) \to 0.$$

Since k is algebraically closed, we have $T \simeq \mathbb{G}_m^t$ and $\#_nT(k) = n^t$. By [Mum14, p.60] or [Mil86, Theorem 8.2], the map $n\colon A \to A$ is finite étale of degree n^{2a}. So $\#_nA(k) = n^{2a}$. Hence, we obtain $\#_nG(k) = \#_nT(k) \cdot \#_nA(k) = n^{t+2a}$. $\square$

Lemma 2.2.8. *Let G be a semi-abelian variety over a field k and let H be a semi-abelian subvariety of G. Let a (resp. a_0) be the dimension of the abelian part and let t (resp. t_0) be the dimension of the toric part of G (resp. H). Then we have*

$$a \geq a_0 \text{ and } t \geq t_0.$$

Proof. We may and do assume that k is algebraically closed. By Remark 2.2.4 (1), we can write the closed immersion $i\colon H \hookrightarrow G$ as the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r] \arrow[d, "j"'] & H \arrow[r] \arrow[d, "i"'] & B \arrow[r] \arrow[d, "f"'] & 0 \\ 0 \arrow[r] & T \arrow[r] & G \arrow[r] & A \arrow[r] & 0, \end{tikzcd}$$

where S and T are tori, and A and B are abelian varieties. Using the snake lemma and noting that i is a closed immersion, we obtain that j is a closed immersion and that the connecting homomorphism $\delta\colon \ker(f) \to \mathrm{coker}(j)$ is a closed immersion. The fact that j is a closed immersion implies that $t \geq t_0$. Note that the reduced connected component $\ker(f)^0_{\mathrm{red}}$ of $\ker(f)$ is a smooth connected closed subgroup[1] of the abelian variety A. Using Lemma 2.2.5 (2), we get that $\mathrm{coker}(j)$ is a torus and $\ker(f)^0_{\mathrm{red}}$ is an abelian variety. By Lemma 2.2.1, the composition of closed immersions

$$\ker(f)^0_{\mathrm{red}} \hookrightarrow \ker(f) \overset{\delta}{\hookrightarrow} \mathrm{coker}(j)$$

[1]Here, we used [DG70, Chapitre II, §5, Corollaire 2.3] that if G is a group scheme locally of finite type over a perfect field k, then G_{red} is a subgroup scheme of G. We shall show that for connected subgroups of semi-abelian varieties, it is unnecessary to assume k to be perfect. See Proposition 2.2.11.

is trivial. Thus $\ker(f)^0_{\mathrm{red}}$ is trivial and $\ker(f)$ has dimension 0, which implies that $a_0 \le a$. □

We will reduce some problems about semi-abelian varieties to relevant ones of finite étale group schemes by using the following result.

Proposition 2.2.9. *Let G be a semi-abelian variety over a field k. Then the collection of closed subschemes $\{{}_nG\}_{n\ge 1,\mathrm{char}(k)\nmid n}$ is (topologically) dense in G, where ${}_nG$ is the kernel of multiplication by n on G.*

Proof. It suffices to prove the assertion for $k = \overline{k}$ and from now on, we assume that k is algebraically closed. Let $X \subset G(k)$ be the union of all ${}_nG(k)$, where $\mathrm{char}(k) \nmid n$ and let H be the reduced closed subscheme of G whose underlying space is the Zariski closure of X. Then by our construction, it is easy to verify that H is a subgroup scheme of G.

The connected component H^0 of the unit is a smooth connected subgroup of the semi-abelian variety G. So H^0 is also semi-abelian by Lemma 2.2.5. Let N be the number of connected components of H. Let a (resp. a_0) be the dimension of the abelian part and let t (resp. t_0) be the dimension of the toric part of G (resp. H^0). By Lemma 2.2.7, we have that

$$\#_nG(k) = n^{2a+t} \quad \text{and} \quad \#_nH^0(k) = n^{2a_0+t_0},$$

the second one of which implies $\#_nH(k) \le Nn^{2a_0+t_0}$. By construction, H contains all torsion points of G of order prime to $\mathrm{char}(k)$. So $\#_nH(k) = \#_nG(k) = n^{2a+t}$. Now, we have $n^{2a+t} \le Nn^{2a_0+t_0}$ for every positive integer n prime to $\mathrm{char}(k)$. Taking n very large, we obtain $2a+t \le 2a_0+t_0$. By Lemma 2.2.8, we have $a \ge a_0$ and $t \ge t_0$. Thus

$$a = a_0 \quad \text{and} \quad t = t_0.$$

So the irreducible variety G has the same dimension as the closed subvariety H^0. Hence $H^0 = G$. □

Lemma 2.2.10. *Let H be a group scheme over a field k. If H_{red} is geometrically reduced, then H_{red} is a closed subgroup scheme of H.*

Proof. The argument of [DG70, Chapitre II, §5, Corollaire 2.3] works here. For readers' convenience, we repeat it: Since H_{red} is geometrically reduced, the scheme $H_{\mathrm{red}} \times_k H_{\mathrm{red}}$ is reduced. Thus the restriction of the multiplication law $m\colon H \times_k H \to H$ to $H_{\mathrm{red}} \times_k H_{\mathrm{red}}$ factors through $H_{\mathrm{red}} \hookrightarrow H$:

$$H_{\mathrm{red}} \times_k H_{\mathrm{red}} \xrightarrow{m_{\mathrm{red}}} H_{\mathrm{red}}.$$

Similarly, the unit morphism and the inverse morphism of H induce morphisms on H_{red}, and it follows that $(H_{\mathrm{red}}, m_{\mathrm{red}})$ is a closed subgroup of (H, m). □

Proposition 2.2.11. *Let G be a semi-abelian variety over a field k and H be a connected closed subgroup of G. Then*

(1) *the collection of closed subschemes $\{{}_nH\}_{n\ge 1,\mathrm{char}(k)\nmid n}$ is (topologically) dense in H;*
(2) *H_{red} is geometrically reduced;*
(3) *H_{red} is a semi-abelian subvariety of G.*

Proof. Let n be a positive integer with $\mathrm{char}(k) \nmid n$. By Lemma 2.2.6, the commutative group scheme ${}_nG$ is finite étale over k. Since ${}_nH$ is a closed subgroup of ${}_nG$, it is also finite étale over k. Let H_0 be the reduced closed subscheme of G whose underlying space is the Zariski closure of $\bigcup_{\mathrm{char}(k)\nmid n} {}_nH$. By [EGA IV$_3$, Corollaire 11.10.7], $(H_0)_{\overline{k}}$ is the reduced closed subscheme of

$G_{\overline{k}}$ whose underlying space is the Zariski closure of $\bigcup_{\mathrm{char}(k)\nmid n} {}_nH_{\overline{k}}$. Since $(H_{\overline{k}})_{\mathrm{red}}$ is a smooth connected closed subgroup of $G_{\overline{k}}$, it is a semi-abelian subvariety of $G_{\overline{k}}$. By Proposition 2.2.9, the collection $\{{}_n((H_{\overline{k}})_{\mathrm{red}})\}_{\mathrm{char}(k)\nmid n}$ is topologically dense in $(H_{\overline{k}})_{\mathrm{red}}$. Since $(H_{\overline{k}})_{\mathrm{red}}$ is a closed subgroup of $H_{\overline{k}}$, we have that ${}_n((H_{\overline{k}})_{\mathrm{red}})$ is a closed subgroup of ${}_nH_{\overline{k}}$, which implies that as topological spaces

$$(H_{\overline{k}})_{\mathrm{red}} \subseteq (H_0)_{\overline{k}} \subseteq H_{\overline{k}}.$$

Because $H_{\overline{k}}$ and $(H_{\overline{k}})_{\mathrm{red}}$ have the same underlying topological space, the above three topological spaces are the same. Thus

$$\dim H_0 = \dim (H_0)_{\overline{k}} = \dim H_{\overline{k}} = \dim H.$$

Because H is irreducible, we conclude that $H_0 = H$ as topological spaces, which completes the proof of the first assertion.

Clearly, $H_0 = H_{\mathrm{red}}$ as schemes. Since $(H_0)_{\overline{k}}$ is reduced, the scheme H_{red} is geometrically reduced.

By Lemma 2.2.10, H_{red} is a closed subgroup of H. Since H_{red} is a smooth connected closed subgroup of the semi-abelian variety G, it is a semi-abelian subvariety by Lemma 2.2.5. □

2.3. **Base change and descent of Deligne 1-motives.**

In the case of primary field extensions, we show that the extension of scalars of Deligne 1-motives is fully faithful. We will also prove some descent results on Deligne 1-motives.

Definition 2.3.1 ([Del74, 10.1.10])**.** (1) A Deligne 1-motive over k is a complex of commutative group schemes

$$M = [L \xrightarrow{u} G],$$

where L is a lattice and G is a semi-abelian variety.

(2) A morphism of Deligne 1-motives from $M = [L \xrightarrow{u} G]$ to $M' = [L' \xrightarrow{u'} G']$ is a commutative square

$$\begin{tikzcd} L \arrow[r, "u"] \arrow[d, "f"'] & G \arrow[d, "g"] \\ L' \arrow[r, "u'"] & G' \end{tikzcd}$$

in the category of group schemes. Denote by $(f,g)\colon M \to M'$ such a morphism.

(3) Denote the category of Deligne 1-motives over k by $\mathsf{M}_1(k)$.

Definition 2.3.2. (1) An extension of fields K/k is called primary if the algebraic closure of k in K is purely inseparable over k.

(2) An extension of fields K/k is called regular if K/k is separable and k is algebraically closed in K.

Remark 2.3.3. Clearly, regular extensions are primary. If k is perfect, then any primary extension of k is automatically regular.

Let K/k be a field extension, K_s be a separable closure of K and k_s be the separable closure of k in K_s. Denote $\mathrm{Gal}(K_s/Kk_s)$ by Γ.

$$\begin{tikzcd}
 & K_s \arrow[d, dash, "\Gamma"] & \\
 & Kk_s \arrow[dl, dash] \arrow[dr, dash] & \\
K \arrow[uur, dashed, dash, bend left, "\mathrm{Gal}(K_s/K)"] \arrow[dr, dash] & & k_s \arrow[dl, dash] \\
 & K\cap k_s \arrow[d, dash] & \\
 & k \arrow[uur, dashed, dash, bend right, "\mathrm{Gal}(k_s/k)"'] &
\end{tikzcd}$$

If K/k is primary, then $K\cap k_s = k$ and the restriction map

$$\pi\colon\ \mathrm{Gal}(K_s/K) \twoheadrightarrow \mathrm{Gal}(Kk_s/K) \xrightarrow{\simeq} \mathrm{Gal}(k_s/k), \quad \sigma \mapsto \sigma|_{Kk_s} \mapsto \sigma|_{k_s}$$

is continuous and surjective with kernel Γ.

Proposition 2.3.4. *Let K/k be a primary extension.*

(1) *The extension of scalars $\pi^*\colon {}^t\mathsf{M}_0(k) \to {}^t\mathsf{M}_0(K)$ is fully faithful. The same result holds for M_0 and* Tori.

(2) *The extension of scalars $\pi^*\colon {}^t\mathsf{M}_0(k) \to {}^t\mathsf{M}_0(K)$ has a right adjoint $\pi_*^{{}^t\mathsf{M}_0}$ and a left adjoint $\pi_\sharp^{{}^t\mathsf{M}_0}$. The same result holds for M_0 and* Tori.

Proof. Thanks Proposition 2.1.4, we identify commutative étale group schemes with the associated discrete Galois modules.

Because the restriction map

$$\pi\colon\ \mathrm{Gal}(K_s/K) \twoheadrightarrow \mathrm{Gal}(Kk_s/K) \xrightarrow{\simeq} \mathrm{Gal}(k_s/k), \quad \sigma \mapsto \sigma|_{Kk_s} \mapsto \sigma|_{k_s}$$

is surjective, it induces a fully faithful functor

$$\begin{aligned} \pi^*\colon {}^t\mathsf{M}_0(k) &\longrightarrow {}^t\mathsf{M}_0(K) \\ L &\longmapsto L, \end{aligned}$$

which corresponds to the extension of scalars of constructible group schemes. The essential image of π^* is the full subcategory of modules on which Γ acts trivially. The functor π^* admits a right adjoint

$$\begin{aligned} \pi_*\colon {}^t\mathsf{M}_0(K) &\longrightarrow {}^t\mathsf{M}_0(k) \\ L &\longmapsto L^\Gamma, \end{aligned}$$

and a left adjoint

$$\begin{aligned} \pi_\sharp\colon {}^t\mathsf{M}_0(K) &\longrightarrow {}^t\mathsf{M}_0(k) \\ L &\longmapsto L_\Gamma, \end{aligned}$$

where $L^\Gamma \subset L$ denotes the submodule of Γ-invariants and $L_\Gamma := L/\langle hx - x \mid h \in \Gamma, x \in L\rangle$ is the space of Γ-coinvariants.

Restricting to lattices, we have a fully faithful functor $\pi^*\colon \mathsf{M}_0(k) \to \mathsf{M}_0(K)$. Since L^Γ is a lattice over k for any $L \in \mathsf{M}_0(K)$, the functor π^* admits a right adjoint $\pi_*\colon L \mapsto L^\Gamma$. On the other hand, for any $L \in \mathsf{M}_0(K)$, the $\mathrm{Gal}(k_s/k)$-module L_Γ can be represented by a unique extension

$$0 \to (L_\Gamma)_{\mathrm{tor}} \to L_\Gamma \to (L_\Gamma)_{\mathrm{fr}} \to 0,$$

where $(L_\Gamma)_{\mathrm{tor}}$ is a finite abelian group with a continuous $\mathrm{Gal}(k_s/k)$-action and $(L_\Gamma)_{\mathrm{fr}}$ is an object of $\mathsf{M}_0(k)$. Since there exists no non-trivial homomorphism from $(L_\Gamma)_{\mathrm{tor}}$ to any $L' \in \mathsf{M}_0(k)$, we can see that

$$\mathrm{Hom}_{\mathsf{M}_0(k)}\left((L_\Gamma)_{\mathrm{fr}}, L'\right) \simeq \mathrm{Hom}_{\mathsf{M}_0(k)}(L_\Gamma, L') \simeq \mathrm{Hom}_{\mathsf{M}_0(K)}(L, \pi^* L').$$

In other words, the functor $L \mapsto (L_\Gamma)_{\mathrm{fr}}$ is the left adjoint of $\pi^*\colon \mathsf{M}_0(k) \to \mathsf{M}_0(K)$.

By Cartier duality (Theorem 2.1.5), we get the assertions for tori from the ones for lattices. $\square$

Denote the category of abelian varieties (resp. semi-abelian varieties) by $\mathsf{AV}(k)$ (resp. $\mathsf{SAV}(k)$), where morphisms are the homomorphisms between group schemes.

Theorem 2.3.5 (Chow)**.** *Let K be a primary field extension of k. Then the extension of scalars*

$$\begin{aligned} \pi^*\colon \mathsf{AV}(k) &\longrightarrow \mathsf{AV}(K), \\ A &\longmapsto A_K \end{aligned}$$

is fully faithful.

Proof. For a modern proof using fpqc descent, see [Con06, Theorem 3.19]. $\square$

Deligne [Del74, 10.2.11–13] defined a self-duality on the category $\mathsf{M}_1(k)$, that he called Cartier duality. Let $M = [L \xrightarrow{u} G]$ be a Deligne 1-motive, and let T and A be the toric and abelian part of G respectively. Then

$$M^\vee = [T^\vee \to G^u],$$

where G^u is an extension of $A^\vee$ by $L^\vee$.

Lemma 2.3.6. *Cartier duality commutes with extension of scalars.*

Proof. Following [BVS01, p.17], we use the symmetric avatar $(L, T^\vee, A, A^\vee, u, v, \psi)$ to denote a Deligne 1-motive $[L \to G]$, where T and A are the toric and abelian part of G respectively. The symmetric avatar of Cartier dual is $(T^\vee, L, A^\vee, A, v, u, \psi^t)$. Since all these $L, T^\vee, A, A^\vee, u, v, \psi, \psi^t$ are compatible with the extension of scalars, we conclude that Cartier duality commutes with the extension of scalars. $\square$

Lemma 2.3.7. *Let $\pi\colon S' \to S$ be a faithfully flat morphism of schemes.*

(1) *The base change functor*

$$\begin{aligned} \pi^*\colon \mathsf{Sch}/S &\longrightarrow \mathsf{Sch}/S', \\ X &\longmapsto \pi^* X := X \times_S S' \end{aligned}$$

is faithful, where Sch/S (resp. Sch/S') is the category of schemes over S (resp. S').

(2) *Assume moreover that π is quasi-compact. If X is a group scheme over S and Z is a closed subscheme of X, then Z is a subgroup scheme of X if and only if $Z_{S'}$ is a subgroup scheme of $X_{S'}$.*

Proof. The first assertion is [EGA IV$_2$, Corollaire 2.2.16], and the second one is part of [Con06, Theorem 3.5]. □

Theorem 2.3.8. *Let K be a primary field extension of k. Then the extension of scalars of Deligne 1-motives*

$$\begin{aligned} \pi^*\colon \mathsf{M}_1(k) &\longrightarrow \mathsf{M}_1(K) \\ [L \to G] &\longmapsto [L_K \to G_K] \end{aligned}$$

is fully faithful.

Proof. By Lemma 2.3.7 (1), it is clear that π^* is faithful. Now, we prove that it is also full. Let $\mathsf{M}_1^{\mathrm{ab}}$ be the full subcategory of M_1 whose objects are $[L \to A]$ with A an abelian variety.

(a) First, we show that the induced functor $\pi^*\colon \mathsf{M}_1^{\mathrm{ab}}(k) \to \mathsf{M}_1^{\mathrm{ab}}(K)$ is full. Consider the morphisms of the form

$$(f,g)\colon [L_K \xrightarrow{u_K} A_K] \longrightarrow [L'_K \xrightarrow{v_K} A'_K]$$

where A and A' are abelian varieties over k. By Proposition 2.3.4 (1) and Theorem 2.3.5, there exist homomorphisms

$$f_0\colon L \to L' \text{ and } g_0\colon A \to A'$$

such that f (resp. g) is the base change of f_0 (resp. g_0). Since (f,g) is a morphism of Deligne 1-motives, we have that

$$(v f_0)_K = v_K f = g u_K = (g_0 u)_K.$$

By Lemma 2.3.7 (1), we obtain that

$$v f_0 = g_0 u.$$

It means that (f_0, g_0) is a morphism of Deligne 1-motives, whose base change is (f,g).

(b) By Lemma 2.3.6, the Cartier duality of M_1 gives an anti-equivalence between $\mathsf{M}_1^{\mathrm{ab}}$ and SAV which commutes with extension of scalars. It follows from (a) that the extension of scalars $\pi^*\colon \mathsf{SAV}(k) \to \mathsf{SAV}(K)$ is fully faithful.

(c) Now, consider $\pi^*\colon \mathsf{M}_1(k) \to \mathsf{M}_1(K)$. Repeating the argument of (a) except for replacing Chow's Theorem 2.3.5 with (b), we conclude that the extension of scalars of Deligne 1-motives is fully faithful. □

Remark 2.3.9. In [Yu19, Theorem 1.2], Yu uses the same strategy as in [Con06, Theorem 3.19] to show that extension of scalars of semi-abelian varieties is fully faithful in the case of primary extension. Our result is a generalization of theirs, and our proof is an alternative to Yu's.

Lemma 2.3.10. *Let K/k be a primary field extension. Let L be a discrete $\mathrm{Gal}(k_s/k)$-module and L' be a discrete $\mathrm{Gal}(K_s/K)$-submodule of L. Then L' is also a discrete $\mathrm{Gal}(k_s/k)$-submodule of L.*

Proof. Since $\mathrm{Gal}(K_s/Kk_s)$ acts trivially on L, its action on the submodule L' is also trivial. Thus L' is a $\mathrm{Gal}(k_s/k)$-submodule of L. □

Proposition 2.3.11. *Let K/k be a primary field extension. Let L be a commutative étale group scheme over k and let L' be a subgroup scheme of L_K. Then the closed immersion $i\colon L' \hookrightarrow L_K$ is defined over k.*

Proof. By the equivalence between discrete Galois modules and commutative étale group schemes (Proposition 2.1.4), this proposition is a reformulation of the above lemma. $\square$

We have a similar result for semi-abelian varieties.

Theorem 2.3.12. *Let K/k be a primary field extension. Let G be a semi-abelian variety over k and H be a semi-abelian subvariety of G_K. Then the closed immersion $i\colon H \hookrightarrow G_K$ of semi-abelian varieties is defined over k.*

Proof. Let n be a positive integer with $\mathrm{char}(k) \nmid n$. By Lemma 2.2.6, the commutative group scheme ${}_nG$ is finite étale over k. Then ${}_nH$ is a finite étale closed subgroup scheme of ${}_nG_K$. By Proposition 2.3.11, the closed immersion ${}_nH \hookrightarrow {}_nG_K$ is the base change of a closed immersion ${}_nH' \hookrightarrow {}_nG$ of finite étale group schemes over k. Let H_0 be the reduced closed subscheme of G whose underlying space is the Zariski closure of the union of such ${}_nH'$'s. By [EGA IV$_3$, Corollaire 11.10.7], $(H_0)_K$ is the reduced closed subscheme of G_K whose underlying space is the Zariski closure of $\bigcup_{\mathrm{char}(k)\nmid n} {}_nH$. Thus we obtain that $(H_0)_K = H$ by Proposition 2.2.11. By Lemma 2.3.7 (2), the subscheme H_0 is a closed subgroup of G. Moreover, the smoothness and connectedness of H descend to H_0. By Lemma 2.2.5, H_0 is a semi-abelian subvariety of G. $\square$

The following lemma is a simple corollary of the above theorem. It is a key ingredient in the construction of Chow image and Chow trace of Deligne 1-motives.

Lemma 2.3.13. *Let K/k be a primary extension. By Proposition 2.3.4, $\pi^*\colon \mathsf{M}_0(k) \to \mathsf{M}_0(K)$ admits a left adjoint $\pi_\sharp^{\mathsf{M}_0}$. Let $(f,g)\colon [L \to G] \longrightarrow [L' \to G']_K$ be a morphism of Deligne 1-motives over K. Then (f,g) factors as*

$$[L \to G] \xrightarrow{(\varepsilon, g_0)} [\pi_\sharp^{\mathsf{M}_0} L \to G_0]_K \longrightarrow [L' \to G']_K,$$

where $\varepsilon\colon L \to (\pi_\sharp^{\mathsf{M}_0} L)_K$ is the unit morphism and $g_0\colon G \to (G_0)_K$ is a surjection.

Proof. By Theorem 2.3.12, the morphism $g\colon G \to G'_K$ factors as

$$G \xrightarrow{g_0} (G_0)_K \xrightarrow{i_K} G'_K,$$

where $(G_0)_K$ is the image of g in the abelian category of commutative algebraic groups over K. Let L_0 be the base change of L' through the closed immersion $i\colon G_0 \hookrightarrow G'$. Then L_0 is a closed subgroup of L' and thus is a lattice by Remark 2.1.3. Using the universal property of fiber products, we can see that the morphism of Deligne 1-motives (f,g) factors as

$$\begin{tikzcd} L \arrow[r, "f_0"] \arrow[d] & (L_0)_K \arrow[r] \arrow[d] & L'_K \arrow[d] \\ G \arrow[r, "g_0"] & (G_0)_K \arrow[r, "i_K"] & G'_K. \end{tikzcd}$$

Then the homomorphism $f_0\colon L \to (L_0)_K$ factors through the unit morphism $\varepsilon\colon L \to (\pi_\sharp^{\mathsf{M}_0} L)_K$, which completes the proof. $\square$

2.4. **Chow image and Chow trace of Deligne 1-motives.**

Lemma 2.4.1. *Let $(P,\leq)$ be a pre-ordered set (i.e., $\leq$ is reflexive and transitive, but not necessarily anti-symmetric) satisfying the following conditions:*

1. *P is filtered, i.e., for $x, y \in P$, there exists an element $z \in P$ with $x \leq z$ and $y \leq z$;*
2. *there exists a subset $Q \subset P$ such that*
   (a) *for any $x \in P$, there exists $y \in Q$ with $x \leq y$;*
   (b) *Q has a maximal element m in the following sense: if $t \in Q$ and $m \leq t$, then $t \leq m$.*

Then m is an upper bound of P, i.e., $x \leq m$ for all $x \in P$.

Proof. For any $x \in P$, there exists an element $y \in P$ with $m \leq y$ and $x \leq y$ by (1). Then there is an element $z \in Q$ such that $y \leq z$ by (2a). Thus $m \leq z$. It follows from (2b) that $z \leq m$. Hence $x \leq m$. □

Theorem 2.4.2. *Let K/k be a primary extension. Then the extension of scalars $\pi^*\colon \mathsf{M}_1(k) \to \mathsf{M}_1(K)$ has a left adjoint $\pi_\sharp^{\mathsf{M}_1}$ and a right adjoint $\pi_*^{\mathsf{M}_1}$, called Chow image and Chow trace respectively.*

Proof. Let $M = [L \to G]$ be a Deligne 1-motive over K. Let P be the set of morphisms of Deligne 1-motives $\varphi\colon M \to \pi^* N$ with N a Deligne 1-motive over k. Let Q be the subset consisting of morphisms of the form

$$(\varepsilon, g)\colon [L \to G] \longrightarrow [\pi_\sharp^{\mathsf{M}_0} L \to G']_K,$$

with $g\colon G \to G'_K$ surjective. Consider the following pre-order: $\varphi' \leq \varphi$ if there exists a morphism $\psi\colon N \to N'$ such that $\varphi' = \pi^*\psi \circ \varphi$.

The pre-ordered set $(P,\leq)$ satisfies all the conditions in Lemma 2.4.1:

(1) Let $\varphi_1\colon M \to \pi^* N_1$ and $\varphi_2\colon M \to \pi^* N_2$ be two elements in P. Then the induced map

$$(\varphi_1, \varphi_2)\colon M \to (N_1)_K \times (N_2)_K \simeq (N_1 \times N_2)_K$$

is a supremum of these two morphisms.

(2) (a) This is Lemma 2.3.13.
(b) Consider the set of closed subgroup schemes of G which are of the form $\ker(g)$ for some (ε, g) in Q. Because G is a noetherian scheme, this set has a minimal element m with respect to inclusion. Say $G/m = G'_K$, where G' is a semi-abelian variety over k. Then the corresponding morphism $(\varepsilon, g)\colon [L \to G] \to [\pi_\sharp^{\mathsf{M}_0} L \to G']_K$ is a maximal element of Q.

Hence by Lemma 2.4.1, the set P has an upper bound which is an element of Q. In other words, there exists a Deligne 1-motive N over k and a morphism $\varphi\colon M \to \pi^* N$ such that for every morphism $\varphi'\colon M \to \pi^* N'$ there exists a morphism $\psi\colon N \to N'$ such that $\varphi' = \pi^*\psi \circ \varphi$. Note that π^* is fully faithful by Theorem 2.3.8, and that such φ has the form (ε, g) with ε the unit morphism and g a surjection. Thus the morphisms ψ satisfying $\varphi' = \pi^*\psi \circ \varphi$ is unique. It means that the functor π^* has a left adjoint $\pi_\sharp^{\mathsf{M}_1}$.

By Lemma 2.3.6, Cartier duality for Deligne 1-motives commutes with extension of scalars. Thus the existence of the right adjoint is obvious by dualizing $\pi_\sharp(M^\vee)$ and using the dual of its universal morphism. □

Corollary 2.4.3. *Let K/k be a primary field extension and $M \in \mathsf{M}_1(k)$. Then we have the following canonical isomorphisms*

$$\pi_\sharp^{\mathsf{M}_1}\pi^* M \xrightarrow{\sim} M$$

and

$$M \xrightarrow{\sim} \pi_*^{\mathsf{M}_1}\pi^* M.$$

Proof. This result holds because π^* is fully faithful according to Theorem 2.3.8. □

The functors $\pi_\sharp^{\mathsf{M}_1}$ and $\pi_*^{\mathsf{M}_1}$ recover some classical constructions, such as Chow's image and trace of abelian varieties. From now on, we sometimes write them simply as $\pi_\sharp$ and π_* respectively.

Corollary 2.4.4. *Let K be a primary field extension of k.*

(1) *For any $L \in \mathsf{M}_0(K)$, we have*

$$\pi_\sharp([L \to 0]) = [\pi_\sharp^{\mathsf{M}_0}(L) \to 0] \quad \textit{and} \quad \pi_*([L \to 0]) = [\pi_*^{\mathsf{M}_0}(L) \to 0],$$

where $\pi_\sharp^{\mathsf{M}_0}$ and $\pi_*^{\mathsf{M}_0}$ are left and right adjoints to extension of scalars of lattices in Proposition 2.3.4.

(2) *The extension of scalars $\pi^*\colon \mathsf{SAV}(k) \to \mathsf{SAV}(K)$ has a left adjoint $\pi_\sharp^{\mathsf{SAV}}$ and a right adjoint π_*^{SAV}. Moreover, for any $G \in \mathsf{SAV}(K)$, we have*

$$\pi_\sharp([0 \to G]) = [0 \to \pi_\sharp^{\mathsf{SAV}}(G)] \quad \textit{and} \quad \pi_*([0 \to G]) = [0 \to \pi_*^{\mathsf{SAV}}(G)].$$

(3) *The extension of scalars $\pi^*\colon \mathsf{AV}(k) \to \mathsf{AV}(K)$ has a left adjoint $\pi_\sharp^{\mathsf{AV}}$ and a right adjoint π_*^{AV}. Moreover, for any $A \in \mathsf{AV}(K)$, we have*

$$\pi_\sharp([0 \to A]) = [0 \to \pi_\sharp^{\mathsf{AV}}(A)] \quad \textit{and} \quad \pi_*([0 \to A]) = [0 \to \pi_*^{\mathsf{AV}}(A)].$$

(4) *The extension of scalars $\pi^*\colon \mathsf{Tori}(k) \to \mathsf{Tori}(K)$ has a left adjoint $\pi_\sharp^{\mathsf{Tori}}$ and a right adjoint π_*^{Tori}. Moreover, for any $T \in \mathsf{Tori}(K)$, we have*

$$\pi_\sharp([0 \to T]) = [0 \to \pi_\sharp^{\mathsf{Tori}}(T)] \quad \textit{and} \quad \pi_*([0 \to T]) = [0 \to \pi_*^{\mathsf{Tori}}(T)].$$

(5) *For any $G \in \mathsf{SAV}(K)$, the abelian part of $\pi_\sharp^{\mathsf{SAV}}(G)$ is isomorphic to the Chow image of the abelian part of G, and the torus part of $\pi_*^{\mathsf{SAV}}(G)$ is isomorphic to the Chow trace of the torus part of G.*

Proof. Keep the notations in the proof of Theorem 2.4.2. Then for any Deligne 1-motive M over K, the unit morphism $\varphi\colon M \to \pi^*\pi_\sharp(M)$ is a maximal element of Q. By definition, for $M = [L \to 0]$ with $L \in \mathsf{M}_0(K)$, the elements of Q are the morphisms

$$(\varepsilon, g)\colon [L \to 0] \longrightarrow [\pi_\sharp^{M_0} L \to G']_K,$$

where g is surjective. So $G' = 0$ and Q has a unique element $(\varepsilon, 0)\colon [L \to 0] \to [\pi_\sharp^{\mathsf{M}_0}(L) \to 0]_K$. So

$$\pi_\sharp([L \to 0]) = [\pi_\sharp^{\mathsf{M}_0}(L) \to 0].$$

Similarly, for $M = [0 \to G]$ with $G \in \mathsf{SAV}(K)$, the elements of Q are the morphisms

$$(\varepsilon, g)\colon [0 \to G] \longrightarrow [0 \to G']_K,$$

where g is surjective. Thus the left adjoint $\pi_\sharp(M)$ has the form $[0 \to G_0]$ for some $G_0 \in \mathsf{SAV}(k)$ and the canonical morphism $G \to \pi^* G_0$ is surjective. So $\pi^* \colon \mathsf{SAV}(k) \to \mathsf{SAV}(K)$ has a left adjoint $\pi_\sharp^{\mathsf{SAV}}$ and we have

$$\pi_\sharp([0 \to G]) = [0 \to \pi_\sharp^{\mathsf{SAV}}(G)].$$

By Lemma 2.2.5, if G is an abelian variety (resp. a torus), then $\pi_\sharp^{\mathsf{SAV}}(G)$ is an abelian variety (resp. a torus) because the canonical morphism $G \to \pi^*\pi_\sharp^{\mathsf{SAV}}(G)$ is surjective. Hence $\pi^* \colon \mathsf{AV}(k) \to \mathsf{AV}(K)$ has a left adjoint $\pi_\sharp^{\mathsf{AV}}$, and for an abelian variety A/K, we have that

$$\pi_\sharp([0 \to A]) = [0 \to \pi_\sharp^{\mathsf{AV}}(A)].$$

The same results hold for tori.

By Cartier duality, we can obtain the assertions on the right adjoints.

The last assertion follows from Remark 2.2.4 (1) and the definition of Chow image and Chow trace. □

Remark 2.4.5. The functors $\pi_\sharp^{\mathsf{AV}}$ and π_*^{AV} are Chow's K/k-image and K/k-trace of abelian varieties. So our result recovers Chow's image and trace. This justifies the name of Chow image and Chow trace of Deligne 1-motives.

In the remaining part of this section, we study the Chow image and Chow trace of a "nontrivial" Deligne 1-motive $[L \to A]$ with A an abelian variety. The case when A has trivial Chow trace is quite simple.

Corollary 2.4.6. *Let K/k be a primary field extension. Let A be an abelian variety over K with $\pi_\sharp(A) = 0$ (equivalently $\pi_*(A) = 0$ by* [Con06, Theorem 6.9]*). Then*

$$\pi_\sharp([L \to A]) = [\pi_\sharp L \to 0] \quad \textit{and} \quad \pi_*([L \xrightarrow{u} A]) = [\pi_* \ker(u) \to 0].$$

Proof. Keep the notations in the proof of Theorem 2.4.2. For $[L \to A]$ over K, the elements of Q are the morphisms

$$(\varepsilon, g) \colon [L \to A] \longrightarrow [\pi_\sharp L \to G]_K,$$

where g is surjective. Since $g \colon A \to G_K$ factors through the unit $A \to (\pi_\sharp A)_K$, we get that $g = 0$ by the assumption that $\pi_\sharp(A) = 0$. Thus $G = 0$ and Q has a unique element $(\varepsilon, 0) \colon [L \to A] \to [\pi_\sharp L \to 0]_K$. So

$$\pi_\sharp([L \to A]) = [\pi_\sharp L \to 0].$$

Now, we compute its Chow trace. Write $\pi_*([L \xrightarrow{u} A]) = [L_0 \to G_0]$. For any semi-abelian variety G' over k, we have that

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{SAV}(k)}(G', G_0) &\simeq \mathrm{Hom}_{\mathsf{M}_1(k)}([0 \to G'], [L_0 \to G_0]) \\
&\simeq \mathrm{Hom}_{\mathsf{M}_1(K)}([0 \to G'_K], [\mathbb{Z} \to A]) \\
&\simeq \mathrm{Hom}_{\mathsf{SAV}(K)}(G'_K, A).
\end{aligned}$$

It implies that $G_0 \simeq \pi_*(A) = 0$. For any lattice L' over k, we have that

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{M}_0(k)}(L', L_0) &\simeq \mathrm{Hom}_{\mathsf{M}_1(k)}([L' \to 0], [L_0 \to G_0]) \\
&\simeq \mathrm{Hom}_{\mathsf{M}_1(K)}([L'_K \to 0], [L \xrightarrow{u} A]) \\
&\simeq \mathrm{Hom}_{\mathsf{M}_0(K)}(L'_K, \ker(u)).
\end{aligned}$$

It follows that $L_0 = \pi_* \ker(u)$. □

A more interesting case is when K is the function field of a smooth and geometrically connected variety X/k and A is defined over k. We shall use the Albanese scheme $\mathscr{A}_{X/k}$ [Ram01, §1.3]. By a semi-abelian group scheme, we mean a group scheme locally of finite type over k whose neutral component is a semi-abelian variety. The Albanese scheme $\mathscr{A}_{X/k}$ is the universal semi-abelian group scheme with a morphism $\mathbb{Z}(X) \to \mathscr{A}_{X/k}$, where $\mathbb{Z}(X)$ is the presheaf $U \mapsto \mathbb{Z}[\mathrm{Mor}_k(U,X)]$. The neutral component $\mathscr{A}^0_{X/k}$ is Serre's generalized Albanese semi-abelian variety of X/k. For a fixed rational point $x \in X(k)$, Serre's Albanese semi-abelian variety is the universal one with a morphism $X \to \mathscr{A}^0_{X/k}$ mapping x to 0, which implies that the canonical map

$$\mathrm{Hom}_k(\mathscr{A}_{X/k}, A) \to \mathrm{Hom}_k(\mathscr{A}^0_{X/k}, A)$$

is surjective.

Lemma 2.4.7. *Let k be a field and let K be the function field of a smooth and geometrically connected variety X/k with a rational point $x \in X(k)$. Let A be an abelian variety over k. Then we have isomorphisms*

$$\mathrm{Hom}_k(\mathscr{A}_{X/k}, A) \simeq A(X) \simeq A(K),$$

and

$$\mathrm{Hom}_k(\mathscr{A}^0_{X/k}, A) \simeq A(K)/A(k).$$

In particular,

(1) *$A(K)/A(k)$ is a finitely generated free abelian group;*
(2) *a rational point in $A(K)$ belongs to $A(k)$ if and only if the corresponding homomorphism $\mathscr{A}^0_{X/k} \to A$ is trivial.*

Proof. The isomorphism $\mathrm{Hom}_k(\mathscr{A}_{X/k}, A) \simeq A(X)$ is the definition of Albanese scheme. We have $A(X) \simeq A(K)$ by the valuative criterion of properness and Weil's extension theorem ([BLR90, §4.4, Theorem 1]). Applying $\mathrm{Hom}_k(-, A)$ to the short exact sequence

$$0 \to \mathscr{A}^0_{X/k} \to \mathscr{A}_{X/k} \to \mathbb{Z} \to 0,$$

we get an exact sequence

$$0 \to \mathrm{Hom}_k(\mathbb{Z}, A) \to \mathrm{Hom}_k(\mathscr{A}_{X/k}, A) \to \mathrm{Hom}_k(\mathscr{A}^0_{X/k}, A) \to 0.$$

The first term is isomorphic to $A(k)$ and the second term is isomorphic to $A(K)$. Thus the third term is isomorphic to $A(K)/A(k)$. By Lemma 2.2.1, the third term is also isomorphic to the Hom-group from the abelian part of $\mathscr{A}^0_{X/k}$ to A. Recall [Mil86, Theorem 12.5] that the Hom-groups between abelian varieties are always finitely generated and free. We are done. $\square$

Proposition 2.4.8. *Let k be a field and let K be the function field of a smooth and geometrically connected variety X/k with a rational point $x \in X(k)$. Let A be an abelian variety over k. Then*

$$\pi_*([\mathbb{Z} \xrightarrow{u} A_K]) = \begin{cases} [\mathbb{Z} \xrightarrow{u} A], & \text{if } u(1) \in A(k); \\ [0 \to A], & \text{if } u(1) \notin A(k). \end{cases}$$

and

$$\pi_\sharp([\mathbb{Z} \xrightarrow{u} A_K]) = [\mathbb{Z} \xrightarrow{v} B],$$

where B is the cokernel of $u(1)\colon \mathscr{A}^0_{X/k} \to A$, and $v\colon \mathbb{Z} \to B$ is the k-homomorphism corresponding to the trivial map $\mathscr{A}^0_{X/k} \to B$.

Proof. First, we compute the Chow trace. The case when $u(1) \in A(k)$ is a special case of Corollary 2.4.3. Suppose now $u(1) \notin A(k)$. We claim that morphisms $(f,g)\colon [L_K \xrightarrow{v_K} G_K] \to [\mathbb{Z} \xrightarrow{u} A_K]$ all satisfy $f = 0$. Otherwise, the image of $f\colon L_K \to \mathbb{Z}$ will be $n\mathbb{Z}$ for some positive integer n. By Theorem 2.3.8, the morphism $g\colon G_K \to A_K$ is defined over k, which implies that $nu(1) \in A(k)$. Thus $u(1)$ is a torsion element of the abelian group $A(K)/A(k)$. By Lemma 2.4.7, we get $u(1) \in A(k)$, a contradiction. Now, it is clear from the claim that $[0 \to A]$ is the Chow trace.

Finally, we compute the Chow image. Keep the notations in the proof of Theorem 2.4.2. The elements of Q are the morphisms

$$(\mathrm{id}, g_K)\colon [\mathbb{Z} \xrightarrow{u} A_K] \longrightarrow [\mathbb{Z} \xrightarrow{w_K} G_K].$$

Here $g\colon A \to G$ is a surjective homomorphism; we implicitly used Theorem 2.3.8 to say that g is defined over k. As a quotient of an abelian variety, G must also be an abelian variety. Since $g(u(1)) = w_K(1)$ comes from $w(1) \in G(k)$, the composition

$$\mathscr{A}^0_{X/k} \longrightarrow \mathscr{A}_{X/k} \xrightarrow{u(1)} A \xrightarrow{g} G$$

is trivial by Lemma 2.4.7. Thus g factors through B, which completes the proof. □

3. Direct and inverse images of n-motivic sheaves

In the remaining part of this chapter, we will study the Chow trace of Deligne 1-motives in the framework of Voevodsky's triangulated categories of (étale) motives. In this section, we study Voevodsky's category of homotopy invariant sheaves ([Voe00], [MVW06]) and some subcategories defined in [ABV09]. We are mainly interested in the direct and inverse images of such sheaves.

Throughout this section, k is a field of exponential characteristic p, i.e., $p = 1$ if $\mathrm{char}(k)$ is zero, and $p = \mathrm{char}(k)$ otherwise. Let Λ be the ring $\mathbb{Z}[\frac{1}{p}]$.

3.1. **Presheaves with transfers.** Let Sm/k be the category of smooth separated schemes of finite type over k. Recall Voevodsky's category of finite correspondences [MVW06, Lecture 1]: Given $X, Y \in \mathsf{Sm}/k$, an elementary correspondence from X to Y is an integral closed subschemes W of $X \times Y$ which is finite and surjective over a connected component of X. We denote by $\mathrm{Cor}_k(X,Y)$ the group of finite correspondences, i.e., the free abelian group generated by the elementary correspondences. Given elementary correspondences $V \in \mathrm{Cor}_k(X,Y)$ and $W \in \mathrm{Cor}_k(Y,Z)$, the composition $W \circ V$ is defined to be the pushforward of the intersection product $(V \times Z) \cdot (X \times W)$ of the corresponding cycles in $X \times Y \times Z$, along the projection $p\colon X \times Y \times Z \to X \times Z$. Here, the intersection product and the pushforward of cycles are defined in [Ful98]. See [MVW06, p. 4] for the verification that $W \circ V$ is a finite correspondence from X to Z. Extending this composition linearly, we get the composition of arbitrary finite correspondences, which is associative and bilinear and has Δ_X as the identity of $\mathrm{Cor}_k(X,X)$. Let $\mathsf{Cor}(k)$ be the (additive) category whose objects are the same as Sm/k and whose morphisms from X to Y are elements of $\mathrm{Cor}_k(X,Y)$. The graph of a morphism yields a functor $\gamma_k\colon \mathsf{Sm}/k \to \mathsf{Cor}(k)$. We consider the category $\mathsf{PST}(k,\Lambda)$ of presheaves with transfers of Λ-modules on Sm/k, i.e., the category of additive contravariant functors from $\mathsf{Cor}(k)$ to the category of Λ-modules. For $X \in \mathsf{Sm}/k$, we denote by $\Lambda_{\mathrm{tr}}(X)$ the presheaf with transfers

$$\Lambda_{\mathrm{tr}}(X)(U) := \mathrm{Cor}_k(U,X) \otimes_{\mathbb{Z}} \Lambda.$$

Let K/k be a field extension. Then we have an obvious extension of scalars functor $e\colon \mathsf{Cor}(k) \to \mathsf{Cor}(K)$ taking X to X_K and $Z \in \mathsf{Cor}_k(X,Y)$ to $Z_K \in \mathsf{Cor}_K(X_K, Y_K)$. It induces a direct image functor

$$e_*^{\mathsf{PST}}\colon \mathsf{PST}(K) \to \mathsf{PST}(k),\ \mathscr{F} \mapsto \mathscr{F} \circ e.$$

The functor e_*^{PST} is clearly exact.

Proposition 3.1.1. (1) *The functor e_*^{PST} admits a left adjoint e^*_{PST};*

(2) $e^*_{\mathsf{PST}}(\Lambda_{\mathrm{tr}}(X)) = \Lambda_{\mathrm{tr}}(X_K)$*;*
(3) *The functor e^*_{PST} is exact.*

Proof. Everything is formal except the left exactness in (3). See [Sus17, Proposition 1.1 and Theorem 4.1]. ☐

3.2. **Étale sheaves with transfers.** Recall that a presheaf with transfers $\mathscr{F}$ is called an étale sheaf with transfers if its underlying presheaf $\mathscr{F} \circ \gamma$ is an étale sheaf on Sm/k. We denote by $\mathsf{Shv}_{\text{ét}}(\mathsf{Sm}/k, \Lambda)$ the category of étale sheaves of Λ-modules on Sm/k, and denote by $\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\Lambda)$ the full subcategory of $\mathsf{PST}(k,\Lambda)$ whose objects are the étale sheaves with transfers. By [MVW06, Lemma 6.2], $\Lambda_{\mathrm{tr}}(X)$ is an étale sheaf with transfers.

Proposition 3.2.1. *The category of étale sheaves with transfers has the following properties:*

(1) *The inclusion functor $\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\Lambda) \hookrightarrow \mathsf{PST}(k,\Lambda)$ has an exact left adjoint*

$$a_{\text{ét}}\colon \mathsf{PST}(k,\Lambda) \longrightarrow \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\Lambda).$$

(2) *The category $\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\Lambda)$ is a Grothendieck abelian category generated by the sheaves $\Lambda_{\mathrm{tr}}(X)$.*
(3) *The forgetful functor $\gamma_*\colon \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\Lambda) \to \mathsf{Shv}_{\text{ét}}(\mathsf{Sm}/k,\Lambda)$ is conservative and commutes with all small limits and colimits.*
(4) *The functor γ_* admits a left adjoint $\gamma^*\colon \mathsf{Shv}_{\text{ét}}(\mathsf{Sm}/k,\Lambda) \to \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\Lambda)$.*

Proof. See [MVW06, 6.18 and the proof of 6.19]; see also [CD19, 10.3.3, 10.3.9, 10.3.11]. ☐

We consider some categories introduced in [ABV09]. For $n \in \mathbb{N}$, we denote by $(\mathsf{Sm}/k)_{\leq n}$ the full subcategory of Sm/k whose objects are the smooth schemes over k of dimension less than or equal to n. Similarly, we denote by $\mathsf{Cor}(k_{\leq n})$ the full subcategory of $\mathsf{Cor}(k)$ having the same objects as $(\mathsf{Sm}/k)_{\leq n}$. We consider the Λ-additive dual $\mathsf{PST}(k_{\leq n},\Lambda)$ of $(\mathsf{Sm}/k)_{\leq n}$. As above, we have the notion of étale sheaves with transfers on $(\mathsf{Sm}/k)_{\leq n}$. We denote by $\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k_{\leq n},\Lambda)$ the full subcategory of $\mathsf{PST}(k_{\leq n},\Lambda)$ whose objects are the étale sheaves with transfers. For $X \in (\mathsf{Sm}/k)_{\leq n}$, we denote by $\Lambda_{\leq n}(X)$ the presheaf with transfers

$$\Lambda_{\leq n}(X)(U) := \mathsf{Cor}_k(U,X) \otimes_{\mathbb{Z}} \Lambda, \text{ where } U \in (\mathsf{Sm}/k)_{\leq n}.$$

Let K/k be a field extension. Then we have the following commutative diagram

$$\begin{tikzcd}
(\mathsf{Sm}/k)_{\le n} \arrow[rr] \arrow[dd] \arrow[dr] & & \mathsf{Sm}/k \arrow[dd] \arrow[dr] & \\
& \mathsf{Cor}(k_{\le n}) \arrow[rr] \arrow[dd] & & \mathsf{Cor}(k) \arrow[dd] \\
(\mathsf{Sm}/K)_{\le n} \arrow[rr] \arrow[dr] & & \mathsf{Sm}/K \arrow[dr] & \\
& \mathsf{Cor}(K_{\le n}) \arrow[rr] & & \mathsf{Cor}(K)
\end{tikzcd}$$

where the vertical arrows are base change functors induced by the morphism $\operatorname{Spec} K \to \operatorname{Spec} k$, the horizontal arrows are inclusions, and the arrows towards the lower right are graph functors. Note that the four functors on the back side are continuous functors for the étale topology in the sense of [SGA 4_{I}, Exposé III, Définition 1.1], i.e., the corresponding direct images of sheaves are still sheaves. So the above diagram induces the following commutative diagram

$$\begin{tikzcd}
\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(K,\Lambda) \arrow[rr] \arrow[dd] \arrow[dr] & & \mathsf{Shv}_{\text{ét}}(\mathsf{Sm}/K,\Lambda) \arrow[dd] \arrow[dr] & \\
& \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(K_{\le n},\Lambda) \arrow[rr] \arrow[dd] & & \mathsf{Shv}_{\text{ét}}((\mathsf{Sm}/K)_{\le n},\Lambda) \arrow[dd] \\
\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\Lambda) \arrow[rr] \arrow[dr] & & \mathsf{Shv}_{\text{ét}}(\mathsf{Sm}/k,\Lambda) \arrow[dr] & \\
& \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k_{\le n},\Lambda) \arrow[rr] & & \mathsf{Shv}_{\text{ét}}((\mathsf{Sm}/k)_{\le n},\Lambda)
\end{tikzcd}$$

where the vertical arrows e_* are direct images of sheaves, the horizontal arrows γ_* are forgetful functors, and the arrows towards the lower right σ_{n*} are 'restriction functors'. In particular, $\Lambda_{\le n}(X) = \sigma_{n*}\Lambda_{\mathrm{tr}}(X)$ is an étale sheaf with transfers for $X \in (\mathsf{Sm}/k)_{\le n}$. Noting that the inclusion functor $(\mathsf{Sm}/k)_{\le n} \hookrightarrow \mathsf{Sm}/k$ is also co-continuous. Thus σ_{n*} is exact.

Lemma 3.2.2 ([ABV09, Lemma 1.1.12])**.** *The functor* $\sigma_{n*}\colon \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\Lambda) \to \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k_{\le n},\Lambda)$ *has a left adjoint*

$$\begin{aligned}
\sigma_n^*\colon \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k_{\le n},\Lambda) &\longrightarrow \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\Lambda)\\
\mathscr{F} &\longmapsto \varinjlim_{\Lambda_{\le n}(X)\to\mathscr{F}} \Lambda_{\mathrm{tr}}(X),
\end{aligned}$$

where the colimit is computed in $\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\Lambda)$.

Definition 3.2.3 ([ABV09, Definition 1.1.13])**.** An étale sheaf with transfers $\mathscr{F} \in \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\Lambda)$ is said to be strongly n-generated if the co-unit

$$\sigma_n^*\sigma_{n*}\mathscr{F} \to \mathscr{F}$$

is an isomorphism. Denote by $\mathsf{Shv}^{\mathrm{tr}}_{\le n}(k,\Lambda)$ the category of strongly n-generated étale sheaves.

Lemma 3.2.4 ([ABV09, Lemma 1.1.17])**.** *The functor σ_n^* is fully faithful and induces an equivalence between* $\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k_{\le n},\Lambda)$ *and* $\mathsf{Shv}^{\mathrm{tr}}_{\le n}(k,\Lambda)$.

Lemma 3.2.5. (1) *The direct image functor* $e^{\mathrm{tr}}_*\colon \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(K,\Lambda) \to \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\Lambda)$ *has a left adjoint*

$$e^*_{\mathrm{tr}}\colon \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\Lambda) \longrightarrow \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(K,\Lambda)$$

and $e_{\mathrm{tr}}^*\Lambda_{\mathrm{tr}}(X) \simeq \Lambda_{\mathrm{tr}}(X_K)$.

(2) *The direct image functor* $e_*^{\leq n}\colon \mathsf{Shv}_{\text{ét}}^{\mathrm{tr}}(K_{\leq n},\Lambda) \to \mathsf{Shv}_{\text{ét}}^{\mathrm{tr}}(k_{\leq n},\Lambda)$ *has a left adjoint*

$$e_{\leq n}^*\colon \mathsf{Shv}_{\text{ét}}^{\mathrm{tr}}(k_{\leq n},\Lambda) \longrightarrow \mathsf{Shv}_{\text{ét}}^{\mathrm{tr}}(K_{\leq n},\Lambda)$$

and $e_{\leq n}^*(\Lambda_{\leq n}(X)) = \Lambda_{\leq n}(X_K)$.

(3) *We have the following natural isomorphism*

$$\sigma_n^* \circ e_{\leq n}^* \simeq e_{\mathrm{tr}}^* \circ \sigma_n^*.$$

Proof. (1) Clearly, $a_{\text{ét}} \circ e_{\mathsf{PST}}^*$ is left adjoint to e_*^{tr}, where $a_{\text{ét}}$ is the functor in Proposition 3.2.1(1). We have

$$a_{\text{ét}}(e_{\mathsf{PST}}^*(\Lambda_{\mathrm{tr}}(X))) = a_{\text{ét}}(\Lambda_{\mathrm{tr}}(X_K)) = \Lambda_{\mathrm{tr}}(X_K),$$

where the second equality holds because $\Lambda_{\mathrm{tr}}(X_K)$ is already an étale sheaf with transfers.

(2) The proof is similar to (1).

(3) This is a direct corollary of $\sigma_{n*} \circ e_*^{\mathrm{tr}} = e_*^{\leq n} \circ \sigma_{n*}$. □

Proposition 3.2.6. *The inverse image functor* $e_{\mathrm{tr}}^*\colon \mathsf{Shv}_{\text{ét}}^{\mathrm{tr}}(k,\Lambda) \to \mathsf{Shv}_{\text{ét}}^{\mathrm{tr}}(K,\Lambda)$ *is exact.*

Proof. Note that $e_{\mathrm{tr}}^* = a_{\text{ét}} \circ e_{\mathsf{PST}}^*$. By Proposition 3.1.1 (3) and Proposition 3.2.1, the functors e_{PST}^* and $a_{\text{ét}}$ are both exact. Thus e_{tr}^* is also exact. □

3.3. **Homotopy invariant sheaves.**

Definition 3.3.1 ([MVW06, Definitions 2.15 and 9.22])**.** Let k be a field.

(1) A presheaf with transfers $\mathscr{F}$ is said to be homotopy invariant if the projection $X \times_k \mathbb{A}_k^1 \to X$ induces an isomorphism

$$\mathscr{F}(X) \xrightarrow{\sim} \mathscr{F}(X \times_k \mathbb{A}_k^1).$$

(2) An étale sheaf with transfers $\mathscr{F}$ is said to be strictly homotopy invariant if the projection $X \times_k \mathbb{A}_k^1 \to X$ induces isomorphisms

$$H_{\text{ét}}^i(X,\mathscr{F}) \xrightarrow{\sim} H_{\text{ét}}^i(X \times_k \mathbb{A}_k^1,\mathscr{F}) \text{ for all } i \geq 0.$$

We denote by $\mathsf{HI}_{\text{ét}}(k,\Lambda)$ the full subcategory of $\mathsf{Shv}_{\text{ét}}^{\mathrm{tr}}(k,\Lambda)$ whose objects are homotopy invariant sheaves.

Theorem 3.3.2 (Voevodsky, Suslin)**.** *Let k be a field of exponential characteristic p and let Λ be the ring* $\mathbb{Z}[1/p]$.

(1) *If $\mathscr{F}$ is a homotopy invariant presheaf with transfers, then $a_{\text{ét}}(\mathscr{F})$ is strictly homotopy invariant.*

(2) *The category* $\mathsf{HI}_{\text{ét}}(k,\Lambda)$ *is a thick subcategory of* $\mathsf{Shv}_{\text{ét}}^{\mathrm{tr}}(k,\Lambda)$. *In particular,* $\mathsf{HI}_{\text{ét}}(k,\Lambda)$ *is abelian.*

Proof. The first assertion is essentially due to Voevodsky and Suslin. In fact, Voevodsky established this result for the Nisnevich topology and perfect fields k ([MVW06, Theorem 24.1]), and Suslin generalized Voevodsky's result to arbitrary fields ([Sus17, Theorem 3.4]). Then one can deduce the result for the étale topology by using Suslin's rigidity theorem [MVW06, Theorem 7.20]. See [BVK16, Proposition 1.7.5] and [ABV09, Proposition 1.1.2].

The second assertion follows immediately from the first one and the five lemma. □

Lemma 3.3.3 ([ABV09, Lemmas 1.1.1 and 1.1.2])**.** *The inclusion* $\iota\colon \mathsf{HI}_{\text{ét}}(k,\Lambda) \hookrightarrow \mathsf{Shv}^{\text{tr}}_{\text{ét}}(k,\Lambda)$ *admits a left adjoint*

$$h_0^{\text{ét}}\colon \mathsf{Shv}^{\text{tr}}_{\text{ét}}(k,\Lambda) \to \mathsf{HI}_{\text{ét}}(k,\Lambda).$$

Here, $h_0^{\text{ét}}(\mathscr{F})$ *is given by the étale sheaf with transfers associated with the* 0*-th homology of the Suslin complex* $C_*\mathscr{F}$ *(*[MVW06, Lecture 2]*).*

For $X \in \mathsf{Sm}/k$, we let

$$h_0^{\text{ét}}(X) := h_0^{\text{ét}}(\Lambda_{\text{tr}}(X)).$$

Lemma 3.3.4. *The direct image* e_*^{tr} *maps homotopy invariant sheaves to homotopy invariant sheaves. In other words, we have a functor* $e_*^{\mathsf{HI}}\colon \mathsf{HI}_{\text{ét}}(K,\Lambda) \to \mathsf{HI}_{\text{ét}}(k,\Lambda)$ *such that the following diagram is commutative*

$$\begin{tikzcd} \mathsf{HI}_{\text{ét}}(K,\Lambda) \arrow[r, hook, "\iota_K"] \arrow[d, dotted, "e_*^{\mathsf{HI}}"'] & \mathsf{Shv}^{\text{tr}}_{\text{ét}}(K,\Lambda) \arrow[d, "e_*^{\text{tr}}"] \\ \mathsf{HI}_{\text{ét}}(k,\Lambda) \arrow[r, hook, "\iota_k"'] & \mathsf{Shv}^{\text{tr}}_{\text{ét}}(k,\Lambda). \end{tikzcd}$$

Proof. For $X \in \mathsf{Sm}/k$, we have the following commutative diagram

$$\begin{tikzcd} (K \times_k \mathbb{A}^1_k) \times_K (K \times_k X) \arrow[r, "\sim"] \arrow[dr, "g"'] & K \times_k (\mathbb{A}^1_k \times_k X) \arrow[d, "f"] \\ & K \times_k X, \end{tikzcd}$$

where f is the base change of the projection $\mathbb{A}^1_k \times_k X \to X$ and g is the projection to $K \times_k X$. For $\mathscr{F} \in \mathsf{HI}_{\text{ét}}(K,\Lambda)$, the morphism $\mathscr{F}(g)$ is an isomorphism. Thus $\mathscr{F}(f)$ is also an isomorphism, i.e., the morphism $(e_*^{\text{tr}}\mathscr{F})(X) \to (e_*^{\text{tr}}\mathscr{F})(\mathbb{A}^1_k \times_k X)$ induced by the projection is an isomorphism, which means that $e_*^{\text{tr}}\mathscr{F}$ is homotopy invariant. □

Lemma 3.3.5 ([Sus17, Proposition 4.9])**.** *The functor* e^*_{PST} *preserves homotopy invariant presheaves with transfers.*

Proposition 3.3.6. *The inverse image* e^*_{tr} *maps homotopy invariant sheaves to homotopy invariant sheaves. In other words, we have a functor* $e^*_{\mathsf{HI}}\colon \mathsf{HI}_{\text{ét}}(k,\Lambda) \to \mathsf{HI}_{\text{ét}}(K,\Lambda)$ *such that the following diagram is commutative*

$$\begin{tikzcd} \mathsf{HI}_{\text{ét}}(k,\Lambda) \arrow[r, hook, "\iota_k"] \arrow[d, dotted, "e^*_{\mathsf{HI}}"'] & \mathsf{Shv}^{\text{tr}}_{\text{ét}}(k,\Lambda) \arrow[d, "e^*_{\text{tr}}"] \\ \mathsf{HI}_{\text{ét}}(K,\Lambda) \arrow[r, hook, "\iota_K"'] & \mathsf{Shv}^{\text{tr}}_{\text{ét}}(K,\Lambda). \end{tikzcd}$$

Moreover, the functor e^*_{HI} *is exact.*

Proof. Note that $e^*_{\text{tr}} = a_{\text{ét}} \circ e^*_{\mathsf{PST}}$. By Lemma 3.3.5 and Theorem 3.3.2 (1), the functors e^*_{PST} and $a_{\text{ét}}$ preserve homotopy invariance. Thus e^*_{tr} maps homotopy invariant sheaves to homotopy invariant sheaves.

By Proposition 3.2.6, the functor e^*_{tr} is exact. Note that ι_k and ι_K are both exact by Theorem 3.3.2 (2). Thus the functor e^*_{HI} is exact. □

Corollary 3.3.7. (1) *The functor* e^*_{HI} *is left adjoint to* e^{HI}_*.

(2) *The unit* $\mathrm{id} \to \iota_k h^{\text{ét}}_0$ *induces a natural isomorphism*

$$h^{\text{ét}}_0 e^*_{\mathrm{tr}} \xrightarrow{\sim} h^{\text{ét}}_0 e^*_{\mathrm{tr}} \iota_k h^{\text{ét}}_0 = e^*_{\mathsf{HI}} h^{\text{ét}}_0.$$

In particular, $e^*_{\mathsf{HI}}(h^{\text{ét}}_0(X)) \simeq h^{\text{ét}}_0(X_K)$.

Proof. It is clear that e^*_{HI} is left adjoint to e^{HI}_* by the adjunction $(e^*_{\mathrm{tr}}, e^{\mathrm{tr}}_*)$. Taking left adjoint to $\iota_k \circ e^{\mathsf{HI}}_* = e^{\mathrm{tr}}_* \circ \iota_K$, we get the second assertion. □

The following result is established by Suslin [Sus17] for the Nisnevich topology and the Zariski topology. We deal with the étale topology.

Proposition 3.3.8. *Let k be a field of characteristic $p > 0$ and let K/k be a purely inseparable extension. Then*

(1) *the functor* $e^{\mathsf{PST}}_* : \mathsf{PST}(K, \Lambda) \to \mathsf{PST}(k, \Lambda)$ *is an equivalence of categories with quasi-inverse* e^*_{PST};
(2) *the functor* $e^{\mathrm{tr}}_* : \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(K, \Lambda) \to \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k, \Lambda)$ *is an equivalence of categories with quasi-inverse* $e^*_{\mathrm{tr}} = e^*_{\mathsf{PST}}$;
(3) *the functor* $e^{\mathsf{HI}}_* : \mathsf{HI}^{\mathrm{tr}}_{\text{ét}}(K, \Lambda) \to \mathsf{HI}^{\mathrm{tr}}_{\text{ét}}(k, \Lambda)$ *is an equivalence of categories with quasi-inverse* e^*_{HI}.

Proof. (1) This is [Sus17, Corollary 1.14].

(2) First, assume that K is a perfect closure of k. It suffices to show that if $\mathscr{G}$ is an étale sheaf, then $e^*_{\mathsf{PST}}\mathscr{G}$ is also an étale sheaf. Since $\mathscr{G} \simeq e^{\mathsf{PST}}_* e^*_{\mathsf{PST}}\mathscr{G}$ by (1), it suffices to check that if $\mathscr{F}$ is a presheaf with transfers over K such that $e^{\mathsf{PST}}_*\mathscr{F}$ is an étale sheaf with transfers, then $\mathscr{F}$ itself is also an étale sheaf. Suppose that $U \in \mathsf{Sm}/K$ and $\{V_i \to U\}$ is an étale covering. We check the sheaf condition for $\mathscr{F}$. It suffices clearly to deal with the case when U is irreducible. Then by [Sus17, Lemma 1.12], there exist an irreducible $X \in \mathsf{Sm}/k$ and a finite surjective purely inseparable[2] morphism $U \to X_K$. By [SGA 4_{II}, Exposé VIII, Théorème 1.1], the étale morphisms $V_i \to U$ descend to étale morphisms $Y_i \to X_K$, and then descend to étale morphisms $Z_i \to X$, i.e., we have the following Cartesian squares:

$$\begin{tikzcd}
V_i \arrow[r] \arrow[d] & Y_i \arrow[r] \arrow[d] & Z_i \arrow[d] \\
U \arrow[r] & X_K \arrow[r] \arrow[d] & X \arrow[d] \\
 & \operatorname{Spec} K \arrow[r] & \operatorname{Spec} k.
\end{tikzcd}$$

By our assumption, $e^{\mathsf{PST}}_*\mathscr{F}$ is a sheaf on $(\mathsf{Sm}/k)_{\text{ét}}$. Thus we have the following exact sequence

$$0 \to \mathscr{F}(X_K) \to \prod \mathscr{F}(Y_i) \to \prod \mathscr{F}(Y_i \times_{X_K} Y_j).$$

Using [Sus17, Lemma 2.4], we obtain the following exact sequence from the above one

$$0 \to \mathscr{F}(U) \to \prod \mathscr{F}(V_i) \to \prod \mathscr{F}(V_i \times_U V_j),$$

[2] Purely inseparable morphisms are also called radicial or universally injective morphisms in other literatures.

which means that $\mathscr{F}$ is a sheaf.

For a general purely inseparable extension K/k, a perfect closure k^{perf} of k is also a perfect closure of K. Thus the inverse image functor from k to k^{perf} is an equivalence and so is the inverse image functor from K to k^{perf}. It follows that the inverse image functor from k to K is also an equivalence.

(3) The last assertion is a combination of Lemma 3.3.4, Proposition 3.3.6 and the second assertion. □

3.4. n-motivic sheaves.

Definition 3.4.1 ([ABV09, Definition 1.1.20])**.** A homotopy invariant sheaf $\mathscr{F} \in \mathsf{HI}_{\text{ét}}(k,\Lambda)$ is said to be n-motivic if the natural morphism

$$h_0^{\text{ét}}\sigma_n^*\sigma_{n*}\mathscr{F} \longrightarrow h_0^{\text{ét}}(\mathscr{F}) = \mathscr{F}$$

is an isomorphism. We denote by $\mathsf{HI}_{\le n}(k,\Lambda)$ the full subcategory of n-motivic sheaves.

Remark 3.4.2. In [Ayo11], n-motivic sheaves are called n-presented $\mathcal{H}$-sheaves.

Remark 3.4.3. As explained in [ABV09, Remark 1.1.21], a homotopy invariant sheaf is n-motivic if and only if it is isomorphic to $h_0^{\text{ét}}\sigma_n^*\mathscr{F}$ for some $\mathscr{F} \in \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k_{\le n},\Lambda)$.

Lemma 3.4.4. *We have a pair of adjoint functors*

$$h_0^{\text{ét}}\sigma_n^* \colon \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k_{\le n},\Lambda) \rightleftarrows \mathsf{HI}_{\le n}(k,\Lambda) \colon \sigma_{n*}\iota_n,$$

where $\iota_n \colon \mathsf{HI}_{\le n}(k,\Lambda) \hookrightarrow \mathsf{HI}_{\text{ét}}(k,\Lambda)$ *and* $\iota \colon \mathsf{HI}_{\text{ét}}(k,\Lambda) \hookrightarrow \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\Lambda)$ *are the inclusions. Moreover, the functor*

$$\sigma_{n*}\iota_n \colon \mathsf{HI}_{\le n}(k,\Lambda) \hookrightarrow \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k_{\le n},\Lambda)$$

is fully faithful.

Proof. The adjunction follows from the adjunctions $(h_0^{\text{ét}},\iota)$ and (σ_n^*,σ_{n*}). By definition, the co-unit

$$(h_0^{\text{ét}}\sigma_n^*)(\sigma_{n*}\iota_n)\mathscr{F} \longrightarrow \mathscr{F}$$

is an isomorphism for every $\mathscr{F} \in \mathsf{HI}_{\le n}(k,\Lambda)$, which implies that $\sigma_{n*}\iota_n$ is fully faithful. □

Lemma 3.4.5. *The functor e^*_{HI} maps n-motivic sheaves to n-motivic sheaves. More precisely, there exists a functor $e_n^* \colon \mathsf{HI}_{\le n}(k,\Lambda) \to \mathsf{HI}_{\le n}(K,\Lambda)$ such that the following diagram is commutative*

$$\begin{tikzcd}
\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k_{\le n},\Lambda) \arrow[r, "h_0^{\text{ét}}\sigma_n^*"] \arrow[d, "e^*_{\le n}"'] & \mathsf{HI}_{\le n}(k,\Lambda) \arrow[r, "\iota_n"] \arrow[d, dotted, "e^*_n"'] & \mathsf{HI}_{\text{ét}}(k,\Lambda) \arrow[d, "e^*_{\mathsf{HI}}"] \\
\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(K_{\le n},\Lambda) \arrow[r, "h_0^{\text{ét}}\sigma_n^*"] & \mathsf{HI}_{\le n}(K,\Lambda) \arrow[r, "\iota_n"] & \mathsf{HI}_{\text{ét}}(K,\Lambda).
\end{tikzcd}$$

In particular, for $X \in (\mathsf{Sm}/k)_{\le n}$, we have

$$e_n^*(h_0^{\text{ét}}(X)) \simeq h_0^{\text{ét}}(X_K).$$

Proof. Let $\mathscr{F} \in \mathsf{HI}_{\le n}(k,\Lambda)$. By Remark 3.4.3, we write $\mathscr{F} = h_0\sigma_n^*\mathscr{G}$ for $\mathscr{G} \in \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k_{\le n},\Lambda)$. Then

$$e^*_{\mathsf{HI}}\iota_n\mathscr{F} = e^*_{\mathsf{HI}}\iota_n h_0^{\text{ét}}\sigma_n^*\mathscr{G} \simeq h_0^{\text{ét}} e^*_{\mathrm{tr}}\sigma_n^*\mathscr{G} \simeq h_0^{\text{ét}}\sigma_n^* e^*_{\le n}\mathscr{G},$$

where the second isomorphism holds by Corollary 3.3.7 (2), and the last isomorphism holds by Lemma 3.2.5 (3). Then by Remark 3.4.3 again, $e^*_{\mathsf{HI}}\iota_n\mathscr{F}$ is n-motivic. □

Definition 3.4.6. We call e_n^* the inverse image functor of n-motivic sheaves. We define the direct image functor e_{n*} of n-motivic sheaves to be the composition

$$\mathsf{HI}_{\leq n}(K,\Lambda) \xhookrightarrow{\iota_n} \mathsf{HI}_{\text{ét}}(K,\Lambda) \xrightarrow{e_*^{\mathsf{HI}}} \mathsf{HI}_{\text{ét}}(k,\Lambda) \xrightarrow{h_0^{\text{ét}}\sigma_n^*\sigma_{n*}} \mathsf{HI}_{\leq n}(k,\Lambda).$$

Lemma 3.4.7 ([ABV09, Lemma 1.1.23])**.** *We have natural isomorphisms*

$$(\sigma_{n*}\iota) \xrightarrow{\sim} (\sigma_{n*}\iota)(h_0^{\text{ét}}\sigma_n^*)(\sigma_{n*}\iota) \xrightarrow{\sim} (\sigma_{n*}\iota).$$

Lemma 3.4.8. (1) *The functor $h_0^{\text{ét}}\sigma_n^*\sigma_{n*}\iota$ is right adjoint to $\iota_n\colon \mathsf{HI}_{\leq n} \hookrightarrow \mathsf{HI}_{\text{ét}}$.*

(2) *We have a commutative diagram*

$$\begin{tikzcd}
\mathsf{HI}_{\text{ét}}(K,\Lambda) \arrow[r, "h_0^{\text{ét}}\sigma_n^*\sigma_{n*}\iota"] \arrow[d, "e_*^{\mathsf{HI}}"'] & \mathsf{HI}_{\leq n}(K,\Lambda) \arrow[r, "\sigma_{n*}\iota_n"] \arrow[d, "e_{n*}"'] & \mathsf{Shv}_{\text{ét}}^{\text{tr}}(K_{\leq n},\Lambda) \arrow[d, "e_*^{\leq n}"] \\
\mathsf{HI}_{\text{ét}}(k,\Lambda) \arrow[r, "h_0^{\text{ét}}\sigma_n^*\sigma_{n*}\iota"] & \mathsf{HI}_{\leq n}(k,\Lambda) \arrow[r, "\sigma_{n*}\iota_n"] & \mathsf{Shv}_{\text{ét}}^{\text{tr}}(k_{\leq n},\Lambda).
\end{tikzcd}$$

(3) *The functor e_{n*} is right adjoint to e_n^*.*

Proof. It is easy to check the first two assertions by using Lemma 3.4.7. Then the third assertion can be checked easily by using the first assertion. □

3.5. **0-motivic sheaves.** In this subsection, we consider sheaves on the small étale site $(\mathsf{Et}/k)_{\text{ét}} = (\mathsf{Sm}/k)_{\leq 0,\text{ét}}$.

The obvious inclusion of sites $\sigma\colon (\mathsf{Et}/k)_{\text{ét}} \to (\mathsf{Sm}/k)_{\text{ét}}$ is continuous and co-continuous and preserves fiber products and the final object $\operatorname{Spec} k$. Thus it gives an adjunction of categories

$$\sigma^*\colon \mathsf{Shv}_{\text{ét}}(\mathsf{Et}/k,\Lambda) \leftrightarrows \mathsf{Shv}_{\text{ét}}(\mathsf{Sm}/k,\Lambda)\colon \sigma_*,$$

where $\sigma_*\mathscr{F} = \mathscr{F}\circ\sigma$. These two functors are both exact and σ^* is fully faithful. See Lemma A.2.1 for more details. By Theorem 3.2.1 (4), the forgetful functor $\gamma_*\colon \mathsf{Shv}_{\text{ét}}^{\text{tr}}(k,\Lambda) \to \mathsf{Shv}_{\text{ét}}(\mathsf{Sm}/k,\Lambda)$ has a left adjoint γ^*. We have the following result.

Lemma 3.5.1. (1) *The functor*

$$\gamma^*\sigma^*\colon \mathsf{Shv}_{\text{ét}}(\mathsf{Et}/k,\Lambda) \longrightarrow \mathsf{Shv}_{\text{ét}}^{\text{tr}}(k,\Lambda)$$

is exact and fully faithful.

(2) *We have an equivalence*

$$\gamma_*\colon \mathsf{Shv}_{\text{ét}}^{\text{tr}}(k_{\leq 0},\Lambda) \xrightarrow{\sim} \mathsf{Shv}_{\text{ét}}(\mathsf{Et}/k,\Lambda)$$

with quasi-inverse $\sigma_{0*}\gamma^*\sigma^*$, where $\sigma_{0*}\colon \mathsf{Shv}_{\text{ét}}^{\text{tr}}(k,\Lambda) \to \mathsf{Shv}_{\text{ét}}^{\text{tr}}(k_{\leq 0},\Lambda)$ is the restriction functor.

Proof. The first assertion is [CD16, Proposition 3.1.4]. Thus $\gamma^*\sigma^*$ induces an equivalence between $\mathsf{Shv}_{\text{ét}}(\mathsf{Et}/k,\Lambda)$ and the category $\mathsf{Shv}_{\leq 0}^{\text{tr}}(k,\Lambda)$ in Definition 3.2.3 with quasi-inverse $\sigma_*\gamma_*$. Then the second assertion is clear by Lemma 3.2.4. □

By [DG70, Chapitre I, §4, Proposition 6.5], for a scheme X locally of finite type over a field k, there exists an étale k-scheme $\pi_0(X)$ and a morphism $q_X\colon X \to \pi_0(X)$ satisfying the following universal property: for any morphism $f\colon X \to Y$ from X to an étale k-scheme Y, there exists a unique $g\colon \pi_0(X) \to Y$ such that $f = g \circ q_X$. Moreover, the morphism q_X is fully faithful and its fibers are the connected components of X.

Thus we have a functor

$$\pi_0\colon \mathsf{Sm}/k \longrightarrow (\mathsf{Sm}/k)_{\leq 0},$$

which is left adjoint to the inclusion functor. As usual, for a presheaf $\mathscr{F}$ on $(\mathsf{Sm}/k)_{\leq 0}$, the presheaf $\mathscr{F} \circ \pi_0$ on Sm/k will be denoted by $\pi_{0*}(\mathscr{F})$. The following result is stated and used in [ABV09, 1.2.1]. We give a proof here for completeness.

Lemma 3.5.2. *For a sheaf $\mathscr{F}$ on $(\mathsf{Sm}/k)_{\leq 0,\text{ét}}$, the presheaf $\pi_{0*}(\mathscr{F})$ is a sheaf on $(\mathsf{Sm}/k)_{\text{ét}}$.*

Proof. Let U be a smooth variety and let $\{V_i \to U\}$ be an étale covering. Then $\{\pi_0(V_i) \to \pi_0(U)\}$ is an étale covering of $\pi_0(U)$. We claim that $\pi_0(V_i \times_U V_j) \to \pi_0(V_i) \times_{\pi_0(U)} \pi_0(V_j)$ is surjective. Consider the following commutative diagram

$$\begin{tikzcd}
\mathscr{F}(\pi_0(U)) \arrow[r] \arrow[d, equal] & \prod_i \mathscr{F}(\pi_0(V_i)) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, equal] & \prod_{i,j} \mathscr{F}(\pi_0(V_i) \times_{\pi_0(U)} \pi_0(V_j)) \arrow[d, hook] \\
\mathscr{F}(\pi_0(U)) \arrow[r] & \prod_i \mathscr{F}(\pi_0(V_i)) \arrow[r, shift left] \arrow[r, shift right] & \prod_{i,j} \mathscr{F}(\pi_0(V_i \times_U V_j)).
\end{tikzcd}$$

Since $\mathscr{F}$ is a sheaf, the first row is exact and the last vertical arrow is an injection. It follows that the second row is also exact, which means that $\pi_{0*}(\mathscr{F})$ is a sheaf.

Now, we prove the claim: for étale morphisms $V_1 \to U$ and $V_2 \to U$, the canonical morphism

$$\varphi\colon \pi_0(V_1 \times_U V_2) \longrightarrow \pi_0(V_1) \times_{\pi_0(U)} \pi_0(V_2)$$

is surjective. Since π_0 commutes with field extensions ([DG70, Chapitre I, §4, Proposition 6.7]) and commutes with disjoint unions, we may assume that k is separably closed, and V_1, V_2 and U are connected. Then it suffices to show that $V_1 \times_U V_2$ is nonempty. Since the morphism $f_i\colon V_i \to U$ is étale, it is an open mapping. Because U is connected and is smooth over k, it is irreducible. So the intersection of two open subsets $f_1(V_1) \cap f_2(V_2)$ is nonempty, i.e., $f_1(V_1) \times_U f_2(V_2) \neq \varnothing$. Since the canonical morphism

$$V_1 \times_U V_2 \longrightarrow f_1(V_1) \times_U f_2(V_2)$$

is surjective, we obtain that $V_1 \times_U V_2$ is nonempty. □

Remark 3.5.3. This lemma means that $\pi_0\colon (\mathsf{Sm}/k)_{\text{ét}} \to (\mathsf{Sm}/k)_{\leq 0,\text{ét}}$ is a continuous functor in the sense of [SGA 4_{I}, Exposé III, Définition 1.1]. But it is not a continuous functor in the sense of [Stacks, Definition 00WV], which is stronger. In general, the canonical morphism $\pi_0(X \times_U V) \to \pi_0(X) \times_{\pi_0(U)} \pi_0(V)$ is not an isomorphism even if $V \to U$ is an étale morphism. For example, we have the following Cartesian square

$$\begin{tikzcd}
\mu_n \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d] \\
\mathbb{G}_m \arrow[r, "n"] & \mathbb{G}_m,
\end{tikzcd}$$

where the morphism $\operatorname{Spec} k \to \mathbb{G}_m$ is the zero section. If $n \geq 2$ is prime to the characteristic of k, then $\mu_n \to \operatorname{Spec} k$ is étale and $\pi_0(\mu_n) \simeq \mu_n \neq \operatorname{Spec}(k)$.

If Z is an elementary correspondence from X to Y, then $\pi_0(Z)$ is an elementary correspondence from $\pi_0(X)$ to $\pi_0(Y)$ by using the canonical isomorphism ([DG70, Chapitre I, §4, Corollaire 6.10])

$$\pi_0(X \times_k Y) \xrightarrow{\sim} \pi_0(X) \times_k \pi_0(Y)$$

and the fact that $q_X \colon X \to \pi_0(X)$ is surjective. So π_0 induces a functor

$$\pi_0^{\mathrm{tr}} \colon \mathsf{Cor}(k) \to \mathsf{Cor}(k_{\le}0)$$

which is compatible with the graph functor:

$$\begin{tikzcd}
\mathsf{Sm}/k \arrow[r, "\pi_0"] \arrow[d, "\gamma"'] & (\mathsf{Sm}/k)_{\le 0} \arrow[d, "\gamma"'] \\
\mathsf{Cor}(k) \arrow[r, "\pi_0^{\mathrm{tr}}"] & \mathsf{Cor}(k_{\le 0}).
\end{tikzcd}$$

The above commutative diagram induces the following one:

$$\begin{tikzcd}
\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k_{\le 0}, \Lambda) \arrow[r, "\pi_{0*}^{\mathrm{tr}}"] \arrow[d, "\gamma_*"'] & \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k, \Lambda) \arrow[d, "\gamma_*"'] \\
\mathsf{Shv}_{\text{ét}}((\mathsf{Sm}/k)_{\le 0}, \Lambda) \arrow[r, "\pi_{0*}"] & \mathsf{Shv}_{\text{ét}}(\mathsf{Sm}/k, \Lambda),
\end{tikzcd}$$

where π_{0*} and π_{0*}^{tr} are the direct image functors and γ_* are the forgetful functors.

Corollary 3.5.4. *We have $\sigma^* = \pi_{0*}$ and $\sigma_0^* \simeq \pi_{0*}^{\mathrm{tr}}$.*

Proof. For a sheaf $\mathscr{F} \in \mathsf{Shv}_{\text{ét}}(\mathsf{Et}/k, \Lambda)$, the inverse image $\sigma^*\mathscr{F}$ is the sheaf associated with the presheaf

$$U \longmapsto \varinjlim_V \mathscr{F}(V),$$

where V are the étale schemes over k such that $U \to \operatorname{Spec} k$ factors through them. By the universal property of π_0, the above colimit is in fact $\mathscr{F}(\pi_0(U))$. So the above presheaf is in fact $\pi_{0*}\mathscr{F}$, which is already a sheaf by Lemma 3.5.2. Hence $\sigma^*\mathscr{F} = \pi_{0*}\mathscr{F}$. Using Lemma 3.5.1, we can check that $\gamma_*\sigma_0^* \simeq \sigma^*\gamma_*$, where the latter one is $\pi_{0*}\gamma_* = \gamma_*\pi_{0*}^{\mathrm{tr}}$. Since γ_* is conservative by Proposition 3.2.1 (3), we get $\sigma_0^* \simeq \pi_{0*}^{\mathrm{tr}}$. □

Proposition 3.5.5. (1) *The functor $\gamma^*\sigma^* \colon \mathsf{Shv}_{\text{ét}}(\mathsf{Et}/k, \Lambda) \to \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k, \Lambda)$ induces an equivalence of categories between $\mathsf{Shv}_{\text{ét}}(\mathsf{Et}/k, \Lambda)$ and $\mathsf{HI}_{\le 0}(k, \Lambda)$.*

(2) *The embedding $\mathsf{HI}_{\le 0}(k, \Lambda) \hookrightarrow \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k, \Lambda)$ admits a left adjoint π_0^* with*

$$\pi_0^*(\Lambda_{\mathrm{tr}}(X)) = \Lambda_{\mathrm{tr}}(\pi_0(X)).$$

(3) *The category $\mathsf{HI}_{\le 0}(k, \Lambda)$ is a Serre subcategory of $\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k, \Lambda)$.*

Proof. See Lemma 3.5.1 and [ABV09, 1.2.2, 1.2.5, 1.2.7]. □

From now on, we shall write the functor π_0^* simply as π_0.

3.6. **Derived direct image of 1-motivic sheaves.** To study the unbounded derived functors, we use the descent model structure on chain complexes developed by Cisinski and Déglise.

For readers' convenience, we recall the machinery:

Definition 3.6.1 ([CD09, Definition 2.2])**.** Let $\mathcal{A}$ be a Grothendieck category. Let $\mathcal{G}$ be an essentially small set of objects of $\mathcal{A}$ and $\mathcal{H}$ be a subset of $C(\mathcal{A})$. For $E \in \mathcal{G}$, let $D(E)$ be the complex $[E \xrightarrow{\mathrm{id}} E]$, concentrated in degrees 0 and 1, and let $f_E \colon E[-1] \to D(E)$ be the map given by the identity in degree 1.

(1) A morphism in $C(\mathcal{A})$ is defined to be a $\mathcal{G}$-cofibration if it is contained in the smallest class of maps in $C(\mathcal{A})$ closed under pushouts, transfinite compositions and retracts, generated by the morphisms $f_E[n]$ for any integer n and any E in $\mathcal{G}$.
(2) A chain complex $C \in C(\mathcal{A})$ is said to be $\mathcal{G}$-local if for any E in $\mathcal{G}$ and any integer n, there is a canonical isomorphism
$$\mathrm{Hom}_{K(\mathcal{A})}(E[n], C) \xrightarrow{\sim} \mathrm{Hom}_{D(\mathcal{A})}(E[n], C).$$
(3) An object C of $C(\mathcal{A})$ is said to be $\mathcal{H}$-flasque if for any integer n and any H in $\mathcal{H}$,
$$\mathrm{Hom}_{K(\mathcal{A})}(H, C[n]) = 0.$$
(4) The pair $(\mathcal{G}, \mathcal{H})$ is called a descent structure on $\mathcal{A}$ if
   (a) elements in $\mathcal{H}$ are $\mathcal{G}$-cofibrant acyclic complexes;
   (b) every $\mathcal{H}$-flasque complex is $\mathcal{G}$-local.

Theorem 3.6.2 ([CD09, Theorem 2.5])**.** *Let $\mathcal{A}$ be a Grothendieck category endowed with a descent structure $(\mathcal{G}, \mathcal{H})$. Then the category $C(\mathcal{A})$ is a proper cellular model category with quasi-isomorphisms as weak equivalences, and $\mathcal{G}$-cofibrations as cofibrations. Furthermore, a complex $C \in C(\mathcal{A})$ is fibrant if and only if it is $\mathcal{H}$-flasque, or equivalently, $\mathcal{G}$-local.*

Definition 3.6.3 ([CD09, p. 228])**.** Let $\mathcal{A}$ and $\mathcal{A}'$ be two Grothendieck categories. Suppose that $(\mathcal{G}, \mathcal{H})$ (resp. $(\mathcal{G}', \mathcal{H}')$) is a descent structure on $\mathcal{A}$ (resp. $\mathcal{A}'$). A functor $f^* \colon \mathcal{A}' \to \mathcal{A}$ is said to satisfy descent (with respect to the above descent structures) if it satisfies the following conditions:

(1) the functor f^* commutes with small colimits, or equivalently, it has a right adjoint f_*;
(2) $f^*(E')$ is a direct sum of elements of $\mathcal{G}$ for any E' in $\mathcal{G}'$;
(3) $f^*(H')$ is in $\mathcal{H}$ for any H' in $\mathcal{H}'$.

Theorem 3.6.4 ([CD09, Theorem 2.14])**.** *If $f^* \colon \mathcal{A}' \to \mathcal{A}$ satisfies descent, then the pair of adjoint functors*
$$f^* \colon C(\mathcal{A}') \rightleftarrows C(\mathcal{A}) \colon f_*$$
is a Quillen adjunction with respect to the descent model structure. In particular, the functors f^* and f_* have the functors
$$Lf^* \colon D(\mathcal{A}') \to D(\mathcal{A}) \quad \text{and} \quad Rf_* \colon D(\mathcal{A}) \to D(\mathcal{A}')$$
as left and right derived functors respectively, and Lf^* is left adjoint to Rf_*.

Remark 3.6.5. Let $\mathcal{A}$, $\mathcal{A}'$ and $\mathcal{A}''$ be three Grothendieck categories endowed with descent structures. Let $f'^* \colon \mathcal{A}'' \to \mathcal{A}'$ and $f^* \colon \mathcal{A}' \to \mathcal{A}$ be two functors satisfying descent, and let f'_* and f_* be their right adjoints. Then it follows easily from general abstract nonsense about

Quillen adjunctions and the preceding theorem that we have canonical isomorphisms of total derived functors

$$Lf^* \circ Lf'^* \simeq L(f^* \circ f'^*) \quad \text{and} \quad R(f'_* \circ f_*) \simeq Rf'_* \circ Rf_*.$$

In fact, under the above assumptions, the functor f_* preserves fibrant objects and fibrant resolutions compute right derived functors. The condition that f^* satisfies descent can be viewed as an unbounded generalization of the condition that $f_* : \mathcal{A} \to \mathcal{A}'$ preserves injective objects, or flasque sheaves in sheaf theory.

We come back to motivic sheaves.

Lemma 3.6.6. *Let K/k be a field extension. Then we have the following commutative diagram*

$$\begin{tikzcd} D(\mathsf{Shv}_{\text{ét}}((\mathsf{Sm}/k)_{\leq n}, \Lambda)) \arrow[r, "L\gamma^*"] \arrow[d, "Le^*_{\leq n}"'] & D(\mathsf{Shv}^{\text{tr}}_{\text{ét}}(k_{\leq n}, \Lambda)) \arrow[r, "L\sigma^*_n"] \arrow[d, "Le^*_{\leq n}"'] & D(\mathsf{Shv}^{\text{tr}}_{\text{ét}}(k, \Lambda)) \arrow[d, "Le^*_{\text{tr}}"'] \\ D(\mathsf{Shv}_{\text{ét}}((\mathsf{Sm}/K)_{\leq n}, \Lambda)) \arrow[r, "L\gamma^*"] & D(\mathsf{Shv}^{\text{tr}}_{\text{ét}}(K_{\leq n}, \Lambda)) \arrow[r, "L\sigma^*_n"] & D(\mathsf{Shv}^{\text{tr}}_{\text{ét}}(K, \Lambda)). \end{tikzcd}$$

Proof. The non-derived version of this commutative diagram can be found in §3.2. In [CD09, Example 2.3], using Verdier's computation of hypercohomology [SGA 4_{II}, Exposé V, §7], Cisinski and Déglise showed that there is a descent structure $(\mathcal{G}, \mathcal{H})$ on $\mathsf{Shv}_{\text{ét}}((\mathsf{Sm}/k)_{\leq n}, \Lambda)$, where $\mathcal{G}$ is the essentially small family consisting of sheaves $\Lambda(X)$ with $X \in (\mathsf{Sm}/k)_{\leq n}$ and $\mathcal{H}$ is the family of mapping cones of $\Lambda(Y_\bullet) \to \Lambda(X)$ for any étale hypercover $Y_\bullet \to X$ on the small étale site $X_{\text{ét}}$. By [CD16, Proposition 2.2.3], there are similar model structures on $\mathsf{Shv}^{\text{tr}}_{\text{ét}}(k, \Lambda)$ and $\mathsf{Shv}^{\text{tr}}_{\text{ét}}(k_{\leq n}, \Lambda)$ by replacing $\Lambda(X)$ with $\Lambda_{\text{tr}}(X)$. By definition, all the functors in the above diagram satisfy descent. Then the expected result follows from Remark 3.6.5. □

Recall Voevodsky's triangulated category of effective étale motives over a field[3]: $\mathsf{DM}^{\text{eff}}_{\text{ét}}(k, \Lambda)$ is the homotopy category of the Bousfield localization of $C(\mathsf{Shv}^{\text{tr}}_{\text{ét}}(k, \Lambda))$ with respect to the class of arrows $\Lambda_{\text{tr}}(X \times \mathbb{A}^1)[n] \to \Lambda_{\text{tr}}(X)[n]$ for $X \in \mathsf{Sm}/k$ and $n \in \mathbb{Z}$. By the general theory of Bousfield localizations ([Hir03, 4.3.1]), $\mathsf{DM}^{\text{eff}}_{\text{ét}}(k, \Lambda)$ is the full subcategory of $D(\mathsf{Shv}^{\text{tr}}_{\text{ét}}(k, \Lambda))$ whose objects are the $\mathbb{A}^1$-local complexes (also called motivic complexes), i.e., the complexes C such that

$$\mathrm{Hom}_{D(\mathsf{Shv}^{\text{tr}}_{\text{ét}}(k,\Lambda))}(\Lambda_{\text{tr}}(X), C[m]) \simeq \mathrm{Hom}_{D(\mathsf{Shv}^{\text{tr}}_{\text{ét}}(k,\Lambda))}(\Lambda_{\text{tr}}(\mathbb{A}^1_X), C[m]).$$

Denote by $L_{\mathbb{A}^1}$ the $\mathbb{A}^1$-localization functor, which is left adjoint to the obvious inclusion $\mathsf{DM}^{\text{eff}}_{\text{ét}}(k, \Lambda) \hookrightarrow D(\mathsf{Shv}^{\text{tr}}_{\text{ét}}(k, \Lambda))$.

Definition 3.6.7. (1) Denote $M(X)$ the object $L_{\mathbb{A}^1}(\Lambda_{\text{tr}}(X)[0])$ for $X \in \mathsf{Sm}/k$ and call it the homological motive of X.

(2) Denote by $\mathsf{DM}_{\leq n}(k, \Lambda)$ the localizing subcategory of $\mathsf{DM}^{\text{eff}}_{\text{ét}}(k, \Lambda)$ generated by $M(X)$ for $X \in (\mathsf{Sm}/k)_{\leq n}$. We will call it the triangulated category of n-motives.

[3] In fact, Voevodsky defined and studied the triangulated subcategory $\mathsf{DM}^{\text{eff}}_{-,\text{ét}}$ consisting of complexes that are bounded above over perfect fields (with finite cohomological dimension) in [Voe00], [MVW06]. Unbounded motivic complexes (over a base scheme) were studied in some other places, for example, [ABV09], [Ayo11], and the six-functor formalism in the motivic world (e.g., [Ayo07a], [Ayo07b], [Ayo14], [CD19], [CD16]).

Lemma 3.6.8. *Let K/k be a field extension. Then we have the following commutative diagram*

$$\begin{tikzcd} D(\mathsf{Shv}_{\text{ét}}((\mathsf{Sm}/k)_{\leq n},\Lambda)) \arrow[r, "L\gamma^*"] \arrow[d, "Le^*_{\leq n}"'] & D(\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k_{\leq n},\Lambda)) \arrow[r, "L_{\mathbb{A}^1}\circ L\sigma^*_n"] \arrow[d, "Le^*_{\leq n}"'] & \mathsf{DM}_{\leq n}(k,\Lambda) \arrow[d, "e^*_{\mathsf{DM}}"'] \\ D(\mathsf{Shv}_{\text{ét}}((\mathsf{Sm}/K)_{\leq n},\Lambda)) \arrow[r, "L\gamma^*"] & D(\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(K_{\leq n},\Lambda)) \arrow[r, "L_{\mathbb{A}^1}\circ L\sigma^*_n"] & \mathsf{DM}_{\leq n}(k,\Lambda), \end{tikzcd}$$

where e^*_{DM} maps $M(X)$ to $M(X_K)$.

Proof. By [CD16, 2.2.4], we have the following commutative diagram

$$\begin{tikzcd} D(\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\Lambda)) \arrow[r, "L_{\mathbb{A}^1}"] \arrow[d, "Le^*_{\mathrm{tr}}"'] & \mathsf{DM}^{\mathrm{eff}}_{\text{ét}}(k,\Lambda) \arrow[d, "e^*_{\mathsf{DM}}"'] \\ D(\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(K,\Lambda)) \arrow[r, "L_{\mathbb{A}^1}"] & \mathsf{DM}^{\mathrm{eff}}_{\text{ét}}(K,\Lambda), \end{tikzcd}$$

where e^*_{DM} maps $M(X)$ to $M(X_K)$. By [ABV09, §2.2], $L_{\mathbb{A}^1}\circ L\sigma^*_n\colon D(\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k_{\leq n},\Lambda)) \to \mathsf{DM}^{\mathrm{eff}}_{\text{ét}}(k,\Lambda)$ takes values in the subcategory $\mathsf{DM}_{\leq n}(k,\Lambda) \hookrightarrow \mathsf{DM}^{\mathrm{eff}}_{\text{ét}}(k,\Lambda)$. Then this assertion follows from Lemma 3.6.6. $\square$

Now, we focus on 1-motivic sheaves.

Proposition 3.6.9 ([ABV09, Corollary 1.3.5])**.** *The category $\mathsf{HI}_{\leq 1}(k,\Lambda)$ is a Serre subcategory of $\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\Lambda)$. In particular, the inclusion $u_1\colon \mathsf{HI}_{\leq 1}(k,\Lambda) \hookrightarrow \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\Lambda)$ is exact.*

Corollary 3.6.10. (1) *$\mathsf{HI}_{\leq 0}(k,\Lambda)$ is a Serre subcategory of $\mathsf{HI}_{\leq 1}(k,\Lambda)$.*

(2) *The fully faithful functor $\sigma_{1*}u_1\colon \mathsf{HI}_{\leq 1}(k,\Lambda) \hookrightarrow \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k_{\leq 1},\Lambda)$ is exact.*
(3) *The inverse image functor $e^*_1\colon \mathsf{HI}_{\leq 1}(k,\Lambda) \to \mathsf{HI}_{\leq 1}(K,\Lambda)$ (in Lemma 3.4.5) is exact.*

Proof. By Propositions 3.5.5 and 3.6.9, $\mathsf{HI}_{\leq 0}(k,\Lambda)$ and $\mathsf{HI}_{\leq 1}(k,\Lambda)$ are both Serre subcategories of $\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\Lambda)$, which implies the first assertion.

The second assertion holds because the functors σ_{1*} and u_1 are both exact.

By Propositions 3.6.9 and 3.3.2, the inclusion functor $\iota_1\colon \mathsf{HI}_{\leq 1} \hookrightarrow \mathsf{HI}_{\text{ét}}$ is exact. By Proposition 3.3.6, the inverse image functor for $\mathsf{HI}_{\text{ét}}$ is exact. Then the exactness of e^*_1 follows from the natural isomorphism $\iota_1\circ e^*_1 \simeq e^*_{\mathsf{HI}}\circ\iota_1$ (Lemma 3.4.5). $\square$

Since the functors in the above lemma are exact, they can be derived trivially.

Theorem 3.6.11 ([ABV09, Theorem 2.4.1 and Corollary 2.4.9])**.** *The derived functor*

$$u_1\colon D(\mathsf{HI}_{\leq 1}(k,\Lambda)) \to D(\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\Lambda))$$

is fully faithful, and the essential image is the subcategory $\mathsf{DM}_{\leq 1}(k,\Lambda)$.

The following result reduces the study of higher direct images of 1-motivic sheaves to the study of higher direct images of sheaves on the site $(\mathsf{Sm}/k)_{\leq 1,\text{ét}}$.

Proposition 3.6.12. *The following diagram is commutative*

$$\begin{tikzcd} D(\mathsf{HI}_{\leq 1}(K,\Lambda)) \arrow[r, "\sigma_{1*}u_1"] \arrow[d, "Re_{1*}"'] & D(\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(K_{\leq 1},\Lambda)) \arrow[r, "\gamma_*"] \arrow[d, "Re^{\leq 1}_*"'] & D(\mathsf{Shv}_{\text{ét}}((\mathsf{Sm}/K)_{\leq 1},\Lambda)) \arrow[d, "Re^{\leq 1}_*"'] \\ D(\mathsf{HI}_{\leq 1}(k,\Lambda)) \arrow[r, "\sigma_{1*}u_1"] & D(\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k_{\leq 1},\Lambda)) \arrow[r, "\gamma_*"] & D(\mathsf{Shv}_{\text{ét}}((\mathsf{Sm}/k)_{\leq 1},\Lambda)). \end{tikzcd}$$

Proof. Since e_1^* is exact by Corollary 3.6.10 (3), the functor $e^*_{\mathrm{DM}}\colon \mathsf{DM}_{\leq 1}(k,\Lambda)\to\mathsf{DM}_{\leq 1}(K,\Lambda)$ corresponds to the derived functor

$$e_1^*\colon D(\mathsf{HI}_{\leq 1}(k,\Lambda))\longrightarrow D(\mathsf{HI}_{\leq 1}(K,\Lambda)).$$

Then we get the assertion by taking right adjoint to the one in Lemma 3.6.8 for $n=1$. □

Denote by δ the inclusion functor $\mathsf{HI}_{\leq 0}\hookrightarrow\mathsf{HI}_{\leq 1}$, and denote by θ the obvious inclusion of étale sites $\mathsf{Et}/k\hookrightarrow(\mathsf{Sm}/k)_{\leq 1}$. Then $\theta^*\colon\mathsf{Shv}_{\text{ét}}(\mathsf{Et}/k,\Lambda)\hookrightarrow\mathsf{Shv}_{\text{ét}}((\mathsf{Sm}/k)_{\leq 1},\Lambda)$ is canonically isomorphic to the composition of the following functors

$$\mathsf{Shv}_{\text{ét}}(\mathsf{Et}/k,\Lambda)\overset{\gamma^*\sigma^*}{\simeq}\mathsf{HI}_{\leq 0}(k,\Lambda)\overset{\delta}{\hookrightarrow}\mathsf{HI}_{\leq 1}(k,\Lambda)\overset{\sigma_{1*}u_1}{\hookrightarrow}\mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k_{\leq 1},\Lambda)\xrightarrow{\gamma_*}\mathsf{Shv}_{\text{ét}}((\mathsf{Sm}/k)_{\leq 1},\Lambda).$$

The higher direct images of 0-motivic sheaves are compatible with the higher direct images of 1-motivic sheaves in the following sense:

Theorem 3.6.13. *For $\mathscr{F}\in\mathsf{HI}_{\leq 0}(K,\Lambda)$, we have a canonical isomorphism*

$$\delta R^i e_{0*}\mathscr{F}\xrightarrow{\sim}R^i e_{1*}\delta\mathscr{F},$$

where e_{n*} are the direct images of n-motivic sheaves in Definition 3.4.6. In particular, if K/k is primary, then $R^ie_{1*}\delta\mathscr{F}$ is the 0-motivic sheaf associated with the $\mathrm{Gal}(k_s/k)$-module $H^i(\Gamma,\mathscr{F}_{K_s})$, where $\Gamma=\mathrm{Gal}(K_s/Kk_s)$.

Proof. By Corollary A.2.5, for $\mathscr{G}\in\mathsf{Shv}_{\text{ét}}(\mathsf{Et}/K,\Lambda)$, the base change morphism

$$\theta^*R^ie_*^{\leq 0}\mathscr{G}\longrightarrow R^ie_*^{\leq 1}\theta^*\mathscr{G}$$

is an isomorphism for any i. Then by Proposition 3.6.12, we have

$$\begin{aligned}\gamma_*\sigma_{1*}u_1\delta R^ie_{0*}\gamma^*\sigma^*\mathscr{G}&\simeq\gamma_*\sigma_{1*}u_1\delta\gamma^*\sigma^*R^ie_*^{\leq 0}\mathscr{G}\\&\simeq\theta^*R^ie_*^{\leq 0}\mathscr{G}\\&\simeq R^ie_*^{\leq 1}\theta^*\mathscr{G}\\&\simeq R^ie_*^{\leq 1}\gamma_*\sigma_{1*}u_1\delta\gamma^*\sigma^*\mathscr{G}\\&\simeq\gamma_*\sigma_{1*}u_1R^ie_{1*}\delta\gamma^*\sigma^*\mathscr{G}.\end{aligned}$$

By Proposition 3.2.1 and Lemma 3.4.4, the functor $\gamma_*\sigma_{1*}u_1\colon\mathsf{HI}_{\leq 1}(k,\Lambda)\to\mathsf{Shv}_{\text{ét}}((\mathsf{Sm}/k)_{\leq 1},\Lambda)$ is conservative. Thus we get a canonical isomorphism

$$\delta R^ie_{0*}\gamma^*\sigma^*\mathscr{G}\xrightarrow{\sim}R^ie_{1*}\delta\gamma^*\sigma^*\mathscr{G}.$$

By Proposition 3.5.5 (1), every $\mathscr{F}\in\mathsf{HI}_{\leq 0}(K,\Lambda)$ is of the form $\gamma^*\sigma^*\mathscr{G}$, which completes the proof.

The last assertion follows from the fact that if K/k is primary, then e_{0*} corresponds to the functor $M\mapsto M^\Gamma$, which has been used in the proof of Proposition 2.3.4. □

Corollary 3.6.14. (1) *If $\mathscr{F} \in \mathrm{HI}_{\le 0}(K,\Lambda)$, then $R^i e_{1*}\mathscr{F}$ are 0-motivic sheaves for all $i \ge 0$ and are torsion sheaves for all $i \ge 1$.*

(2) *For $\mathscr{F} \in \mathrm{HI}_{\le 1}(K,\Lambda)$, we have $R^i e_{1*}\mathscr{F}$ are torsion 0-motivic sheaves for all $i \ge 2$.*

Proof. If $\mathscr{F} \in \mathrm{HI}_{\le 0}(K,\Lambda)$, then $R^i e_{1*}\mathscr{F}$ are 0-motivic sheaves for all $i \ge 0$ by Theorem 3.6.13.

For arbitrary $\mathscr{F} \in \mathrm{HI}_{\le 1}(K,\Lambda)$, consider the exact sequence of 1-motivic sheaves

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F} \otimes \mathbb{Q} \to \mathscr{F}'' \to 0.$$

By Suslin's rigidity theorem ([MVW06, Theorem 7.20]), the torsion sheaves $\mathscr{F}'$ and $\mathscr{F}''$ are 0-motivic sheaves. Thus $R^i e_{1*}\mathscr{F}'$ and $R^i e_{1*}\mathscr{F}''$ are 0-motivic sheaves for all i and are in fact torsion sheaves by [SGA 4_{III}, Exposé IX, Proposition 1.2(v)]; see also [Stacks, Lemma 0DDD]. Note that $R^i e_*^{\le 1}(\gamma_*\sigma_{1*}\omega_1\mathscr{F} \otimes \mathbb{Q})$ is the étale sheaf associated with the presheaf

$$\begin{aligned}(\mathrm{Sm}/k)_{\le 1} &\longrightarrow \Lambda\text{-}\mathrm{Mod}\\ X &\longmapsto H^i_{\text{ét}}(X_K, \gamma_*\sigma_{1*}\omega_1\mathscr{F} \otimes \mathbb{Q}).\end{aligned}$$

For $X \in (\mathrm{Sm}/k)_{\le 1}$ and $i \ge 2$,

$$H^i_{\text{ét}}(X_K, \gamma_*\sigma_{1*}\omega_1\mathscr{F} \otimes \mathbb{Q}) \simeq H^i_{\mathrm{Nis}}(X_K, \gamma_*\sigma_{1*}\omega_1\mathscr{F} \otimes \mathbb{Q}) = 0,$$

where the first isomorphism holds by [MVW06, Proposition 14.23], and the second one holds because the Nisnevich cohomological dimension is bounded by the Krull dimension ([MV99, §3.1, Proposition 1.8]). By Proposition 3.6.12, for $i \ge 2$,

$$\gamma_*\sigma_{1*}\omega_1 R^i e_{1*}(\mathscr{F} \otimes \mathbb{Q}) \simeq R^i e_*^{\le 1}(\gamma_*\sigma_{1*}\omega_1\mathscr{F} \otimes \mathbb{Q}) = 0.$$

By Proposition 3.2.1 and Lemma 3.4.4, the functor $\gamma_*\sigma_{1*}\omega_1 : \mathrm{HI}_{\le 1}(k,\Lambda) \to \mathrm{Shv}_{\text{ét}}((\mathrm{Sm}/k)_{\le 1},\Lambda)$ is conservative, which implies that

$$R^i e_{1*}(\mathscr{F} \otimes \mathbb{Q}) = 0 \quad \text{for} \quad i \ge 2.$$

Then split the above exact sequence to two short exact sequences and consider the induced long exact sequences of cohomology. Noting that $\mathrm{HI}_{\le 0}$ is a Serre subcategory of $\mathrm{HI}_{\le 1}$ (Corollary 3.6.10 (1)), we obtain that $R^i e_{1*}\mathscr{F}$ are torsion 0-motivic sheaves for $i \ge 2$.

In particular, if $\mathscr{F}$ is a 0-motivic sheaf, then all the sheaves in the above exact sequence are in fact 0-motivic sheaves and $R^i e_{1*}(\mathscr{F} \otimes \mathbb{Q}) = 0$ for $i \ge 1$. Then the same argument as above shows that $R^i e_{1*}\mathscr{F}$ are torsion 0-motivic sheaves for $i \ge 1$. □

3.7. **Inverse images of semi-abelian varieties.** We consider sheaves defined by commutative group schemes.

Definition 3.7.1. Let G be a commutative group scheme over k. Denote $\underline{G}$ the abelian sheaf on $(\mathrm{Sm}/k)_{\text{ét}}$ defined by G, i.e.,

$$\underline{G}(U) = \mathrm{Mor}_{\mathrm{Sm}/k}(U,G) \quad \text{for} \quad U \in \mathrm{Sm}/k.$$

Denote $\underline{G}_\Lambda = \underline{G} \otimes_{\mathbb{Z}} \Lambda$ the presheaf tensor product, i.e.,

$$\underline{G}_\Lambda(U) = \mathrm{Mor}_{\mathrm{Sm}/k}(U,G) \otimes_{\mathbb{Z}} \Lambda \quad \text{for} \quad U \in \mathrm{Sm}/k.$$

Then $\underline{G}_\Lambda$ is a sheaf of Λ-modules on $(\mathrm{Sm}/k)_{\text{ét}}$.

There are transfer structures on such sheaves.

Lemma 3.7.2 ([SS03, Proof of Lemma 3.2], [Org04, Lemmas 3.1.2 and 3.3.1])**.** *Let G be a commutative group scheme over k. Then $\underline{G}_\Lambda$ has a canonical structure of étale sheaves with transfers, which is functorial. More precisely, there exists a unique étale sheaf with transfers $\underline{G}_\Lambda^{\mathrm{tr}}$ such that*

$$\gamma_* \underline{G}_\Lambda^{\mathrm{tr}} \simeq \underline{G}_\Lambda,$$

where $\gamma_* \colon \mathsf{Shv}_{\text{ét}}^{\mathrm{tr}}(k,\Lambda) \to \mathsf{Shv}_{\text{ét}}(\mathsf{Sm}/k,\Lambda)$ is the forgetful functor. Moreover, if G is a commutative étale group scheme or a semi-abelian variety, then $\underline{G}_\Lambda^{\mathrm{tr}}$ is homotopy invariant.

The aim of this subsection is to show the following result.

Proposition 3.7.3. *Let K/k be a field extension and let G be a commutative étale group scheme or a semi-abelian variety over k. Then*

$$e_{\mathsf{HI}}^*(\underline{G}_\Lambda^{\mathrm{tr}}) \simeq \underline{G_K}_\Lambda^{\mathrm{tr}}.$$

We start with the analogue for sheaves without transfers. Denote e_* (resp. e^*) the direct image (resp. inverse image) functor of sheaves on the smooth-étale sites. The following result is standard and well-known.

Lemma 3.7.4. *Let K/k be a field extension and G be a commutative smooth group scheme over k. Then we have a canonical isomorphism*

$$e^*(\underline{G}_\Lambda) \simeq \underline{G_K}_\Lambda.$$

Proof. In this proof, by abuse of notation, we use Sm/k to mean the category of smooth separated schemes locally of finite type over k rather than the full subcategory of smooth separated schemes of finite type over k, which is used in other places of this chapter. By [SGA 4_{I}, Exposé III, Théorème 4.1], these two categories give the same category of étale sheaves.

For $X \in \mathsf{Sm}/k$, denote $\Lambda(X)$ the étale sheaf associated with the presheaf mapping $U \in \mathsf{Sm}/k$ to the free Λ-module generated by $\mathrm{Mor}_k(U,X)$. By Yoneda's lemma, we have that

$$e^*\Lambda(X) \simeq \Lambda(X_K).$$

Recall the following exact sequence of étale sheaves

$$\Lambda(G\times_k G) \longrightarrow \Lambda(G) \longrightarrow \underline{G}_\Lambda \longrightarrow 0,$$

where the first map sends a generator $[(a_1,a_2)]$ to $[a_1]+[a_2]-[a_1+a_2]$, and the second map sends a generator $[g]$ to g. Then the following commutative diagram with exact rows

$$\begin{tikzcd} e^*\Lambda(G\times_k G) \arrow[r] \arrow[d, "\simeq"] & e^*\Lambda(G) \arrow[r] \arrow[d, "\simeq"] & e^*(\underline{G}_\Lambda) \arrow[r] \arrow[d] & 0 \\ \Lambda(G_K\times_K G_K) \arrow[r] & \Lambda(G_K) \arrow[r] & \underline{G_K}_\Lambda \arrow[r] & 0 \end{tikzcd}$$

gives us the desired isomorphism. ☐

We use the Frobenius and Verschiebung morphisms to deal with the direct images of commutative (flat) group schemes in the case of purely inseparable field extensions.

Lemma 3.7.5. *Let k be a field of characteristic $p > 0$ and let K/k be a purely inseparable extension. Let G be a commutative group scheme locally of finite type over k. Then we have a canonical isomorphism*

$$\underline{G}_\Lambda \simeq e_*(\underline{G_K}_\Lambda),$$

In other words, for $X \in \mathsf{Sm}/k$, there is a canonical isomorphism

$$\mathrm{Mor}_k(X,G)\otimes_{\mathbb{Z}}\Lambda \xrightarrow{\sim} \mathrm{Mor}_K(X_K,G_K)\otimes_{\mathbb{Z}}\Lambda.$$

Proof. We divide the proof into several steps:

(a) Let us start with the case $K = k^{1/p}$. Note that the map $k^{1/p} \to k$, $x \mapsto x^p$ is an isomorphism with inverse $x \mapsto x^{1/p}$. So we reduce to show the isomorphism

$$\mathrm{Mor}_k(X,G)\otimes_{\mathbb{Z}}\Lambda \xrightarrow{\sim} \mathrm{Mor}_k(X^{(p)},G^{(p)})\otimes_{\mathbb{Z}}\Lambda,$$

where $X^{(p)}$ (resp. $G^{(p)}$) is the base change of X (resp. G) along the absolute Frobenius of $\mathrm{Spec}\,k$. By Lemma 2.3.7 (1), this map is injective. Now, we show that it is surjective. Let $f\colon X^{(p)} \to G^{(p)}$ be a morphism of k-schemes. Denote $F_{X/k}\colon X \to X^{(p)}$ (resp. $F_{G/k}\colon G \to G^{(p)}$) the relative Frobenius of X/k (resp. G/k). Note that G is always flat over k. Thus by [SGA 3_{I}, Exposé VII, 4.3], there exists a Verschiebung morphism $V_{G/k}\colon G^{(p)} \to G$ such that

$$V_{G/k}\circ F_{G/k} = p\,\mathrm{id}_G \quad\text{and}\quad F_{G/k}\circ V_{G/k} = p\,\mathrm{id}_{G^{(p)}}.$$

Let f_0 be the composition

$$X \xrightarrow{F_{X/k}} X^{(p)} \xrightarrow{f} G^{(p)} \xrightarrow{V_{G/k}} G,$$

and let $f_0^{(p)}\colon X^{(p)} \to G^{(p)}$ be the base change of $f_0\colon X \to G$. Then

$$f_0^{(p)}\circ F_{X/k} = F_{G/k}\circ f_0 = F_{G/k}\circ V_{G/k}\circ f\circ F_{X/k} = p\circ f\circ F_{X/k},$$

Note that $X^{(p)}$ is reduced (because X is smooth), $G^{(p)}$ is separated over $\mathrm{Spec}\,k$, and $F_{X/k}$ is surjective. By [EGA IV_3, Propositions 11.10.4 and 11.10.1 (d)], we obtain

$$f_0^{(p)} = p\circ f.$$

(b) We deal with the case $K = k^{\mathrm{perf}} = \bigcup_{n\in\mathbb{N}} k^{1/p^n}$, a perfect closure of k. By (a), we have the expected isomorphisms for $K = k^{1/p^n}$. Note that $X \to \mathrm{Spec}\,k$ is quasi-compact and quasi-separated and that $G \to \mathrm{Spec}\,k$ is locally of finite presentation. Thus by [EGA IV_3, Théorème 8.8.2], there is a canonical isomorphism

$$\varinjlim_n \mathrm{Mor}_{k^{1/p^n}}(X_{k^{1/p^n}},G_{k^{1/p^n}}) \xrightarrow{\sim} \mathrm{Mor}_{k^{\mathrm{perf}}}(X_{k^{\mathrm{perf}}},G_{k^{\mathrm{perf}}}),$$

which implies the expected result.

(c) Now, we prove the assertion for general purely inseparable field extensions. Let k^{perf} be a perfect closure of k. Then k^{perf} is also a perfect closure of K because K/k is purely inseparable. By (b), we have the two isomorphisms in the following commutative diagram

$$\begin{tikzcd}
\mathrm{Mor}_k(X,G)\otimes_{\mathbb{Z}}\Lambda \arrow[rr] \arrow[dr, "\sim"'] & & \mathrm{Mor}_K(X_K,G_K)\otimes_{\mathbb{Z}}\Lambda \arrow[dl, "\sim"] \\
& \mathrm{Mor}_{k^{\mathrm{perf}}}(X_{k^{\mathrm{perf}}},G_{k^{\mathrm{perf}}})\otimes_{\mathbb{Z}}\Lambda. &
\end{tikzcd}$$

It follows that the horizontal arrow is also an isomorphism. □

Remark 3.7.6. The above lemma is false before inverting p. For example, let $X = \operatorname{Spec} k$ and $G = \mathbb{G}_a$. Then $\mathrm{Mor}_k(X, G) \simeq k$ and $\mathrm{Mor}_K(X_K, G_K) \simeq K$. The natural inclusion $k \hookrightarrow K$ is not an isomorphism in general.

Lemma 3.7.7. *Let K/k be a field extension and G be a commutative smooth group scheme over k. Then*

$$e_{\mathsf{HI}}^* h_0^{\text{ét}} \gamma^* \underline{G}_\Lambda \simeq h_0^{\text{ét}} \gamma^* \underline{G_K}_\Lambda.$$

Proof. In fact, we have

$$e_{\mathsf{HI}}^* h_0^{\text{ét}} \gamma^* \underline{G}_\Lambda \simeq h_0^{\text{ét}} e_{\mathrm{tr}}^* \gamma^* \underline{G}_\Lambda \simeq h_0^{\text{ét}} \gamma^* e^* \underline{G}_\Lambda \simeq h_0^{\text{ét}} \gamma^* \underline{G_K}_\Lambda,$$

where the first and the last isomorphisms hold by Corollary 3.3.7 (2) and Lemma 3.7.4 respectively, and the second isomorphism is obtained by taking left adjoint to $\gamma_* e_*^{\mathrm{tr}} \simeq e_* \gamma_*$ (part of the commutative diagram before Lemma 3.2.2). □

Lemma 3.7.8. *Let G be a commutative étale group scheme or a semi-abelian variety over k. Then*

$$h_0^{\text{ét}} \gamma^* \underline{G}_\Lambda \simeq \underline{G}_\Lambda^{\mathrm{tr}}.$$

Proof. For a perfect field k, Barbieri-Viale and Kahn [BVK16, Lemma 3.9.2] showed that the composition of the forgetful functors

$$\mathsf{HI}_{\text{ét}}(k, \Lambda) \xrightarrow{\iota} \mathsf{Shv}_{\text{ét}}^{\mathrm{tr}}(k, \Lambda) \xrightarrow{\gamma_*} \mathsf{Shv}_{\text{ét}}(\mathsf{Sm}/k, \Lambda)$$

is fully faithful. Thus we have a natural isomorphism

$$h_0^{\text{ét}} \gamma^* \gamma_* \iota(\underline{G}_\Lambda^{\mathrm{tr}}) \xrightarrow{\sim} \underline{G}_\Lambda^{\mathrm{tr}}.$$

In other words, we have the expected isomorphism over perfect fields.

Now, we extend it to the general field k. Let K be a perfect closure of k. Reformulating Lemma 3.7.5, we have $\gamma_* \underline{G}_\Lambda^{\mathrm{tr}} \simeq \gamma_* e_*^{\mathrm{tr}} \underline{G_K}_\Lambda^{\mathrm{tr}}$. Since γ_* is conservative, we have

$$\underline{G}_\Lambda^{\mathrm{tr}} \simeq e_*^{\mathsf{HI}} \underline{G_K}_\Lambda^{\mathrm{tr}}.$$

Consider the following commutative diagram

$$\begin{tikzcd}
h_0^{\text{ét}} \gamma^* \underline{G}_\Lambda \arrow[rr] \arrow[d, "\simeq"'] & & \underline{G}_\Lambda^{\mathrm{tr}} \arrow[d, "\simeq"] \\
e_*^{\mathsf{HI}} e_{\mathsf{HI}}^* h_0^{\text{ét}} \gamma^* \underline{G}_\Lambda \arrow[r, "\simeq"] & e_*^{\mathsf{HI}} h_0^{\text{ét}} \gamma^* \underline{G_K}_\Lambda \arrow[r, "\simeq"] & e_*^{\mathsf{HI}} \underline{G_K}_\Lambda^{\mathrm{tr}},
\end{tikzcd}$$

Here, the left vertical arrow is an isomorphism by Proposition 3.3.8, and the two horizontal arrows at the bottom are isomorphisms successively by Lemma 3.7.7 and the assertion over perfect fields. Thus the horizontal arrow on top is also an isomorphism. □

Remark 3.7.9. In [AHPL16, Proposition 3.10], they proved using qfh topology that if S is an excellent scheme and G is a commutative smooth group scheme over S, then the co-unit

$$\gamma^* \gamma_* \underline{G}_{\mathbb{Q}}^{\mathrm{tr}} \longrightarrow \underline{G}_{\mathbb{Q}}^{\mathrm{tr}}$$

is an isomorphism of étale sheaves with transfers.

We are now ready to prove the main result of this subsection:

Proof of Proposition 3.7.3. Consider the following commutative diagram

$$\begin{tikzcd} e_{\mathsf{HI}}^* h_0^{\text{ét}} \gamma^* \underline{G}_\Lambda \arrow[r, "\simeq"] \arrow[d, "\simeq"'] & e_{\mathsf{HI}}^*(\underline{G}_\Lambda^{\text{tr}}) \arrow[d] \\ h_0^{\text{ét}} \gamma^* \underline{G_K}_\Lambda \arrow[r, "\simeq"] & \underline{G_K}_\Lambda^{\text{tr}}, \end{tikzcd}$$

where the two horizontal isomorphisms hold by Lemma 3.7.8 and the left vertical isomorphism holds by Lemma 3.7.7. Thus the right vertical arrow is also an isomorphism. □

From now on, we shall write the étale sheaves with transfers $\underline{G}_\Lambda^{\text{tr}}$ as $\underline{G}_\Lambda$, or even as $\underline{G}$, G for simplicity if it does not cause confusion.

3.8. **Direct images of semi-abelian varieties.** In this subsection, we show that the Chow trace of a semi-abelian variety is the "connected component" of its direct image.

Following [ABV09], we call a commutative group scheme G over k a semi-abelian group scheme if its connected component of the identity G^0 is a semi-abelian variety and $\pi_0(G)$ is a constructible group scheme. As explained in [ABV09, comments before 1.3.1 and Corollary 1.3.5], semi-abelian group schemes are 1-motivic sheaves.

Definition 3.8.1 ([ABV09, Definition 1.3.7])**.** A 1-motivic sheaf $\mathscr{F}$ is said to be finitely generated if there exist a semi-abelian group scheme G and an epimorphism $q\colon \underline{G}_\Lambda \to \mathscr{F}$. Moreover, if $\ker(q)$ is finitely generated, then $\mathscr{F}$ is said to be finitely presented.

Proposition 3.8.2 ([ABV09, 1.3.8])**.** (1) *Let $\mathscr{F}$ be a finitely presented 1-motivic sheaf. Then there is a unique and functorial exact sequence*

$$0 \to \underline{\mathscr{L}}_\Lambda \to \underline{\mathscr{G}}_\Lambda \to \mathscr{F} \to 0$$

where $\mathscr{G}$ is a semi-abelian group scheme and $\mathscr{L}$ is a lattice.

(2) *Let $\mathscr{F}$ be a 1-motivic sheaf. Then $\mathscr{F}$ is a filtered colimit of finitely presented 1-motivic sheaves.*

Remark 3.8.3. See also [BVK16, Chapter 3] for some basic properties of finitely presented 1-motivic sheaves.

Corollary 3.8.4 (cf. [BVK16, Proposition 3.3.4] and [ABV09, Theorem 1.3.10])**.** *Let $\mathscr{F}$ be a finitely presented 1-motivic sheaf. Then there exists an exact sequence in $\mathsf{HI}_{\leq 1}(k,\Lambda)$:*

$$0 \to \underline{L}_\Lambda \to \underline{G}_\Lambda \to \mathscr{F} \to \underline{E}_\Lambda \to 0,$$

where L is a lattice, G is a semi-abelian variety and E is a constructible group scheme. Moreover, $\underline{E}_\Lambda = \pi_0(\mathscr{F})$, where $\pi_0\colon \mathsf{Shv}_{\text{ét}}^{\text{tr}}(k,\Lambda) \longrightarrow \mathsf{HI}_{\leq 0}(k,\Lambda)$ is a left adjoint to the inclusion functor (Proposition 3.5.5).

Proof. By Proposition 3.8.2, there is a unique and functorial exact sequence

$$0 \to \underline{\mathscr{L}}_\Lambda \to \underline{\mathscr{G}}_\Lambda \to \mathscr{F} \to 0$$

where $\mathscr{G}$ is a semi-abelian group scheme and $\mathscr{L}$ is a lattice. Let G be the identity component of $\mathscr{G}$ and let $\pi_0(\mathscr{G})$ be the quotient group scheme $\mathscr{G}/\mathscr{G}^0$. Let L and E be the kernel and cokernel of the induced morphism $\mathscr{L} \to \pi_0(\mathscr{G})$ respectively. Applying the snake lemma to the following commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \underline{\mathscr{L}}_\Lambda \arrow[r] \arrow[d] & \underline{\mathscr{G}}_\Lambda \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \underline{\pi_0(\mathscr{G})}_\Lambda \arrow[r] & \underline{\pi_0(\mathscr{G})}_\Lambda \arrow[r] & 0 \arrow[r] & 0, \end{tikzcd}$$

we get the expected exact sequence. Applying π_0 to the top row of the above diagram, we have the following exact sequence

$$\underline{\mathscr{L}}_\Lambda \to \underline{\pi_0(\mathscr{G})}_\Lambda \to \pi_0(\mathscr{F}) \to 0.$$

Thus $\pi_0(\mathscr{F})$ is the cokernel $\underline{E}_\Lambda$. □

Definition 3.8.5. (1) For $\mathscr{F} \in \mathsf{Shv}^{\mathrm{tr}}_{\text{ét}}(k,\Lambda)$, we denote $\mathscr{F}^0 := \ker(\mathscr{F} \to \pi_0(\mathscr{F}))$ and call it the connected component of $\mathscr{F}$.

(2) A sheaf $\mathscr{F}$ is called connected if $\pi_0(\mathscr{F}) = 0$. Denote by $\mathsf{HI}^0_{\leq 1}(k,\Lambda)$ the category of connected 1-motivic sheaves.

Lemma 3.8.6. (1) *Let $\mathscr{F} \in \mathsf{HI}_{\leq 1}(k,\Lambda)$ be a 1-motivic sheaf. Then $\mathscr{F}^0$ is connected. In particular, there is no nontrivial morphism from $\mathscr{F}^0$ to any 0-motivic sheaves.*

(2) *Every connected 1-motivic sheaf $\mathscr{F}$ is a filtered colimit of finitely presented connected 1-motivic sheaves.*

Proof. By Proposition 3.8.2 (2), we write $\mathscr{F}$ as a filtered colimit of finitely presented 1-motivic sheaves $\mathscr{F} = \varinjlim_i \mathscr{F}_i$. Since π_0 commutes with colimits (as a left adjoint) and filtered colimits are exact, we have that $\mathscr{F}^0 = \varinjlim_i \mathscr{F}_i^0$. By Corollary 3.8.4, $\pi_0(\mathscr{F}_i^0) = 0$. Thus $\pi_0(\mathscr{F}^0) = \varinjlim_i \pi_0(\mathscr{F}_i^0) = 0$.

If $\mathscr{F}$ is connected, then $\mathscr{F}^0 \simeq \mathscr{F}$. By the above argument, the sheaf $\mathscr{F}^0$ is a filtered colimit of finitely presented connected 1-motivic sheaves. Thus $\mathscr{F}$ is also. □

Let K/k be a field extension. Then the composition

$$\mathsf{HI}_{\leq 1}(K,\Lambda) \xrightarrow{e_{1*}} \mathsf{HI}_{\leq 1}(k,\Lambda) \xrightarrow{(\cdot)^0} \mathsf{HI}^0_{\leq 1}(k,\Lambda)$$

is right adjoint to the composition

$$\mathsf{HI}^0_{\leq 1}(k,\Lambda) \hookrightarrow \mathsf{HI}_{\leq 1}(k,\Lambda) \xrightarrow{e_1^*} \mathsf{HI}_{\leq 1}(K,\Lambda).$$

Theorem 3.8.7. *Let K/k be a primary field extension and G/K be a semi-abelian variety. Then the connected 1-motivic sheaf $(e_{1*}(\underline{G}_\Lambda))^0$ is represented by the Chow trace π_*G, and the 0-motivic sheaf $\pi_0(e_{1*}(\underline{G}_\Lambda))$ is the sheaf associated with the* $\mathrm{Gal}(k_s/k)$*-Λ-module*

$$\mathrm{LN}(G, Kk_s/k_s)_\Lambda := G(Kk_s)/(\pi_*G)(k_s) \otimes_{\mathbb{Z}} \Lambda.$$

In other words, we have an exact sequence of 1-motivic sheaves

$$0 \to \underline{\pi_*G}_\Lambda \to e_{1*}G \to \mathrm{LN}(G, Kk_s/k_s)_\Lambda \to 0.$$

Proof. By Proposition 3.7.3 and Lemma 3.4.5, we have

$$e_1^*(\underline{\pi_* G}_\Lambda) \simeq \underline{(\pi_* G)_K}_\Lambda.$$

Then the co-unit $\pi^*\pi_* \to \mathrm{id}$ induces a morphism $e_1^*(\underline{\pi_* G}_\Lambda) \to \underline{G}_\Lambda$. It suffices to show that for $\mathscr{F} \in \mathsf{HI}^0_{\le 1}(k, \Lambda)$, there exists a unique morphism $f : \mathscr{F} \to \underline{\pi_* G}_\Lambda$ such that the following diagram commutes

$$\begin{tikzcd} e_1^*(\mathscr{F}) \arrow[d, dotted, "e_1^*(f)"'] \arrow[dr] & \\ e_1^*(\underline{\pi_* G}_\Lambda) \arrow[r] & \underline{G}_\Lambda. \end{tikzcd}$$

By Lemma 3.8.6, we may and do assume that $\mathscr{F}$ is a finitely presented connected 1-motivic sheaf. Let

$$0 \to \underline{L}_\Lambda \xrightarrow{a} \underline{G'}_\Lambda \xrightarrow{b} \mathscr{F} \to 0$$

be the presentation of $\mathscr{F}$ in Corollary 3.8.4. By Corollary 3.6.10 (3), the inverse image e_1^* of 1-motivic sheaves is exact. Thus we get another exact sequence

$$0 \to e_1^*(\underline{L}_\Lambda) \to e_1^*(\underline{G'}_\Lambda) \to e_1^*(\mathscr{F}) \to 0.$$

Denote by $[\underline{L}_\Lambda \to \underline{G'}_\Lambda]$ the complex of 1-motivic sheaves concentrated in degrees 0 and 1. Other similar notations below have a similar meaning. Then

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{HI}^0_{\le 1}(k,\Lambda)}(\mathscr{F}, \underline{\pi_* G}_\Lambda) &\simeq \mathrm{Hom}_{C(\mathsf{HI}_{\le 1}(k,\Lambda))}([\underline{L}_\Lambda \to \underline{G'}_\Lambda], [0 \to \underline{\pi_* G}_\Lambda]) \\
&\simeq \mathrm{Hom}_{\mathsf{M}_1(k)}([L \to G'], [0 \to \pi_* G]) \otimes_{\mathbb{Z}} \Lambda \\
&\simeq \mathrm{Hom}_{\mathsf{M}_1(K)}([L_K \to G'_K], [0 \to G]) \otimes_{\mathbb{Z}} \Lambda \\
&\simeq \mathrm{Hom}_{C(\mathsf{HI}_{\le 1}(K,\Lambda))}([\underline{L_K}_\Lambda \to \underline{G'_K}_\Lambda], [0 \to \underline{G}_\Lambda]) \\
&\simeq \mathrm{Hom}_{C(\mathsf{HI}_{\le 1}(K,\Lambda))}([e_1^*(\underline{L}_\Lambda) \to e_1^*(\underline{G'}_\Lambda)], [0 \to \underline{G}_\Lambda]) \\
&\simeq \mathrm{Hom}_{\mathsf{HI}_{\le 1}(K,\Lambda)}(e_1^*\mathscr{F}, \underline{G}_\Lambda),
\end{aligned}$$

where the first and the last isomorphism hold by the above exact sequences, the second and the fourth isomorphisms hold by [BVK16, 3.3.4 d)], the third isomorphism holds by construction of Chow trace, and the second to the last isomorphisms hold by Proposition 3.7.3.

By the above argument, we have $\underline{\pi_* G}_\Lambda \simeq (e_{1*}\underline{G}_\Lambda)^0$. Consider the exact sequence

$$0 \to \underline{\pi_* G}_\Lambda \to e_{1*}(\underline{G}_\Lambda) \to \pi_0(e_{1*}(\underline{G}_\Lambda)) \to 0.$$

Taking the stalk at the geometric point $\operatorname{Spec} k_s \to \operatorname{Spec} k$, we get

$$0 \to (\pi_* G)(k_s)_\Lambda \to G(Kk_s)_\Lambda \to \pi_0(e_{1*}(\underline{G}_\Lambda))_{k_s} \to 0.$$

Thus $\pi_0(e_{1*}(\underline{G}_\Lambda))$ is the 0-motivic sheaf associated with $\mathrm{LN}(A, Kk_s/k_s)_\Lambda$. □

Corollary 3.8.8. *Let K/k be a finitely generated regular extension and let A be an abelian variety over K. Then the 1-motivic sheaf $e_{1*}(\underline{A}_\Lambda)$ is finitely presented.*

Proof. The Lang-Néron theorem ([Con06, Theorem 7.1]) says that $A(Kk_s)/(\pi_* A)(k_s)$ is a finitely generated abelian group. This means that $\pi_0(e_{1*}(\underline{A}_\Lambda))$ is a finitely presented 1-motivic sheaf. Since every extension of two finitely presented 1-motivic sheaves is still finitely presented, the 1-motivic sheaf $e_{1*}(\underline{A}_\Lambda)$ is finitely presented as well. □

It is well-known that an étale sheaf which is an extension of two separated group schemes is represented by a separated group algebraic space and that separated group algebraic spaces over a field are represented by group schemes. In the special case of the exact sequence in Theorem 3.8.7, we can prove the representability without using algebraic spaces. See Proposition B.0.4. Thus we have the following result.

Corollary 3.8.9. *Let K/k be a primary extension of fields of characteristic 0 and let G be a semi-abelian variety over K. Then the 1-motivic sheaf $e_{1*}G$ is represented by a semi-abelian group scheme over k.*

For an abelian variety A over K, we have gotten some information about $e_{1*}(A)$ and $R^i e_{1*}(A)$ for $i \geq 2$ by Theorem 3.8.7 and Corollary 3.6.14. The following result is about the first direct image. It uses Raynaud's results on torsors under abelian schemes.

Theorem 3.8.10. *Let K/k be a field extension, and let A be an abelian variety over K. Then $R^i e_{1*}(A)$ is a torsion 0-motivic sheaf for $i \geq 1$.*

Proof. We use the same argument as in the proof of Corollary 3.6.14. It suffices to show that $R^1 e_*^{\leq 1}(A \otimes \mathbb{Q}) = 0$, where $e_*^{\leq 1}$ is the direct image functor for étale sheaves on $(\mathsf{Sm})_{\leq 1}$. Note that $R^1 e_*^{\leq 1}(A \otimes \mathbb{Q})$ is the étale sheafification of the presheaf

$$\begin{aligned}(\mathsf{Sm}/k)_{\leq 1} &\longrightarrow \Lambda\text{-}\mathsf{Mod},\\ X &\longmapsto H^1_{\text{ét}}(X_K, A) \otimes \mathbb{Q}.\end{aligned}$$

Since X is noetherian and regular, torsors under the abelian scheme A_X are torsion, i.e., $H^1_{\text{ét}}(X_K, A)$ is a torsion group by [Ray70, Proposition XIII 2.6.(ii) and Proposition XIII 2.3.(ii)]. Thus $H^1_{\text{ét}}(X_K, A) \otimes \mathbb{Q} = 0$, which implies that $R^1 e_*^{\leq 1}(A \otimes \mathbb{Q}) = 0$. □

We can also study $R^1 e_{1*}\mathbb{G}_m$ in some interesting cases.

Theorem 3.8.11. *Let $f\colon X \to \operatorname{Spec} k$ be a smooth projective and geometrically connected variety, and let K be the function field of X. Then we have an exact sequence*

$$\mathrm{Div}^0(X_{k_s}) \to \mathrm{Pic}^0_{X/k} \to R^1 e_{1*}\mathbb{G}_m \to 0,$$

where $\mathrm{Div}^0(X_{k_s})$ is $\mathrm{Gal}(k_s/k)$-module of divisors on X_{k_s} algebraically equivalent to zero, viewed as a locally constant étale sheaf. In particular, we have $R^1 e_*\mathbb{G}_m$ is connected.

Proof. For an affine open subscheme $u\colon U \hookrightarrow X$, we have the following exact sequence

$$\bigoplus_{x \in X^{(1)} \cap (X \setminus U)} \mathbb{Z} \to \mathrm{Pic}_{X/k} \to \mathrm{Pic}_{U/k} \to 0,$$

where $\mathrm{Pic}_{X/k}$ (resp. $\mathrm{Pic}_{U/k}$) is the Picard functor $R^1 f_*\mathbb{G}_{m,X}$ (resp. $R^1(fu)_*\mathbb{G}_{m,U}$). Taking colimit, we obtain

$$\mathrm{Div}_{X/k} \to \mathrm{Pic}_{X/k} \to R^1 e_*\mathbb{G}_m \to 0,$$

where $\mathrm{Div}_{X/k}$ is the locally constant étale sheaf associated with the $\mathrm{Gal}(k_s/k)$-module $\mathrm{Div}(X_{k_s})$. Recall that $\mathrm{Pic}_{X/k}$ is represented by a group scheme, which is an extension of the discrete sheaf $\mathrm{NS}_{X/k}$ by the abelian variety $\mathrm{Pic}^0_{X/k}$ (Picard variety). Applying the snake lemma to the following commutative diagram with exact rows

$$\begin{tikzcd} & \mathrm{Div}_{X/k} \arrow[r] \arrow[d] & \mathrm{Pic}_{X/k} \arrow[r] \arrow[d] & R^1 e_* \mathbb{G}_m \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{NS}_{X/k} \arrow[r, equal] & \mathrm{NS}_{X/k} \arrow[r] & 0 & \end{tikzcd}$$

we obtain the desired result. □

4. The 1-motivic t-structure

In this section, we use Ayoub's way of perverting t-structures to get a new t-structure from the standard one on $D(\mathsf{HI}_{\leq 1})$. This new t-structure will be called the 1-motivic t-structure, and the objects in its heart will be called 1-motives. We shall translate the results on higher direct images of 1-motivic sheaves to results on 1-motives.

4.1. **The abelian category of 1-motives with torsion.** Before we deal with the 1-motivic t-structure, we review in this subsection the construction of the abelian category of 1-motives with torsion. This category was introduced in [BVRS03] (in characteristic 0) and studied in details in [BVK16] over perfect fields. It contains the category of Deligne 1-motives. We shall see in the next subsections that this abelian category can be embedded into the heart of the 1-motivic t-structure.

Let k be a field of exponential characteristic p, i.e., $p = 1$ if char(k) is zero, and $p = \mathrm{char}(k)$ otherwise.

Definition 4.1.1 ([BVK16, C.1])**.** (1) An effective 1-motive with torsion over k is a complex of group schemes

$$M = [L \xrightarrow{u} G],$$

where $L \in {}^t\mathsf{M}_0(k)$ is a constructible group scheme and $G \in \mathsf{SAV}(k)$ is a semi-abelian variety.

(2) An effective morphism of 1-motives with torsion from $M = [L \xrightarrow{u} G]$ to $M' = [L' \xrightarrow{u'} G']$ is a commutative square

$$\begin{tikzcd} L \arrow[r, "u"] \arrow[d, "f"'] & G \arrow[d, "g"] \\ L' \arrow[r, "u'"] & G' \end{tikzcd}$$

in the category of group schemes. Denote by $(f,g)\colon M \to M'$ such a morphism. We will denote by $\mathrm{Hom}_{\mathrm{eff}}(M, M')$ the abelian group of effective morphisms.
(3) We denote the category of effective 1-motives with torsion over k by ${}^t\mathsf{M}_1^{\mathrm{eff}}(k)$.
(4) If $[F \to 0]$ is an effective 1-motive with F a finite étale group scheme, then it is called a torsion 1-motive. The full subcategory of ${}^t\mathsf{M}_1^{\mathrm{eff}}(k)$ consisting of torsion 1-motives is denoted by ${}^t\mathsf{M}_1^{\mathrm{tor}}(k)$.

Remark 4.1.2. If $L \in \mathsf{M}_0(k)$ is a lattice, then $[L \to G]$ is a Deligne 1-motive.

Definition 4.1.3 ([BVK16, C.2.1])**.** An effective morphism of 1-motives with torsion from $M = [L \xrightarrow{u} G]$ to $M' = [L' \xrightarrow{u'} G']$ is called a quasi-isomorphism of 1-motives with torsion if it yields a pullback diagram

$$\begin{tikzcd} 0 \arrow[r] & F \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d, "u"'] & L' \arrow[r] \arrow[d, "u'"'] & 0 \\ 0 \arrow[r] & F \arrow[r] & G \arrow[r] & G' \arrow[r] & 0, \end{tikzcd}$$

where F is a finite étale group.

By [BVK16, C.2.4], the class of quasi-isomorphisms is a left multiplicative system in the sense of [KS06, 7.1.7].

Definition 4.1.4 ([BVK16, C.3.1])**.** The category ${}^t\mathsf{M}_1(k)$ of 1-motives with torsion is the localization of ${}^t\mathsf{M}_1^{\mathrm{eff}}(k)$ with respect to the multiplicative class of quasi-isomorphisms. In other words, the objects of ${}^t\mathsf{M}_1(k)$ are the same as the objects in ${}^t\mathsf{M}_1^{\mathrm{eff}}(k)$, and the Hom-sets are given by the formula

$$\mathrm{Hom}_{{}^t\mathsf{M}_1(k)}(M, M') = \varinjlim_{(\widetilde{M}\to M)\ \text{q.i.}} \mathrm{Hom}_{\mathrm{eff}}(\widetilde{M}, M'),$$

where the colimit is taken over the co-filtrant category of all quasi-isomorphisms $\widetilde{M} \to M$.

Remark 4.1.5. By definition, there exist no nontrivial quasi-isomorphisms to $[L \to 0]$ with $L \in {}^t\mathsf{M}_0(k)$. Thus the natural functor

$$\begin{aligned} {}^t\mathsf{M}_0(k) &\longrightarrow {}^t\mathsf{M}_1(k), \\ L &\longmapsto [L \to 0] \end{aligned}$$

is fully faithful.

Definition 4.1.6. Let $\mathcal{A}$ be an additive category. Then we denote by $\mathcal{A}[1/p]$ the category with the same objects as $\mathcal{A}$ but

$$\mathrm{Hom}_{\mathcal{A}[1/p]}(X, Y) := \mathrm{Hom}_{\mathcal{A}}(X, Y) \otimes_{\mathbb{Z}} \mathbb{Z}[1/p].$$

Now, we introduce a special kind of quasi-isomorphisms and use it to reformulate the category of 1-motives with torsion.

Example 4.1.7. Let $M = [L \to G]$ be an effective 1-motive with torsion over k and let n be an integer with $\mathrm{char}(k) \nmid n$. We consider the multiplication by n on G. By [BLR90, 7.3, Lemma 1 and 2], the group scheme ${}_nG$ is finite étale over k. Let $L^{(n)}$ be the pullback of L along $n\colon G \to G$. Thus we have the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & {}_nG \arrow[r] \arrow[d, equal] & L^{(n)} \arrow[r] \arrow[d] & L \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & {}_nG \arrow[r] & G \arrow[r, "n"] & G \arrow[r] & 0. \end{tikzcd}$$

It is clear that $L^{(n)}$ is a constructible group scheme. Thus we obtain a quasi-isomorphism from $[L^{(n)} \to G]$ to $[L \to G]$. We shall denote the effective 1-motive $[L^{(n)} \to G]$ by $M^{(n)}$ and denote this quasi-isomorphism by $s_n : M^{(n)} \to M$.

Lemma 4.1.8. *Let $(f,g)\colon [L' \to G'] \to [L \to G]$ be a quasi-isomorphism of effective 1-motives with torsion. Let n be the degree of the isogeny $g\colon G' \to G$. If $\mathrm{char}(k) \nmid n$, then there exist morphisms $(f',g')\colon [L^{(n)} \to G] \to [L' \to G']$ and $(f^{(n)},g)\colon [L'^{(n)} \to G'] \to [L^{(n)} \to G]$ such that*

$$s_n = (f,g)\circ(f',g') \quad \textit{and} \quad s_n = (f',g')\circ(f^{(n)},g).$$

Proof. By [BLR90, 7.3, Lemma 5], there exists an isogeny of semi-abelian varieties $g'\colon G \to G'$ such that $g' \circ g = n_{G'}$. Thus $g\circ g'\circ g = g \circ n_{G'} = n_G \circ g$. Since g is an epimorphism, we obtain $g \circ g' = n_G$. By the definition of quasi-isomorphisms, L' is the pullback of L along $g : G' \to G$. Then by the universal property of pullback, there exists a homomorphism $f'\colon L^{(n)} \to L'$ such that the following diagram is commutative:

$$\begin{tikzcd}
L^{(n)} \arrow[r] \arrow[d, dotted, "f'"] \arrow[dd, bend right] & G \arrow[d, "g'"'] \arrow[dd, bend left, "n_G"] \\
L' \arrow[r] \arrow[d, "f"] & G' \arrow[d, "g"'] \\
L \arrow[r] & G
\end{tikzcd}$$

Since the lower square and the composition square are pullbacks, the upper square is also a pullback. Then by the universal property of pullback, there exists a homomorphism $f^{(n)}\colon L'^{(n)} \to L^{(n)}$ such that the following diagram is commutative:

$$\begin{tikzcd}
L'^{(n)} \arrow[r] \arrow[d, dotted, "f^{(n)}"] \arrow[dd, bend right] & G' \arrow[d, "g"'] \arrow[dd, bend left, "n_{G'}"] \\
L^{(n)} \arrow[r] \arrow[d, "f'"] & G \arrow[d, "g'"'] \\
L' \arrow[r] & G',
\end{tikzcd}$$

which completes the proof. □

Proposition 4.1.9. *Let M and M' be 1-motives with torsion over k. Then*

$$\varinjlim_{(\widetilde{M}\to M)\ q.i.} \mathrm{Hom}_{\mathrm{eff}}(\widetilde{M}, M') \otimes \mathbb{Z}[1/p] \simeq \varinjlim_{(n,p)=1} \mathrm{Hom}_{\mathrm{eff}}(M^{(n)}, M') \otimes \mathbb{Z}[1/p].$$

In other words, ${}^t\mathsf{M}_1(k)[1/p]$ is the localization of ${}^t\mathsf{M}_1^{\mathrm{eff}}(k)[1/p]$ with respect to the quasi-isomorphisms of the form s_n for n prime to p.

Proof. By Lemma 4.1.8, the embedding functor from the category of quasi-isomorphisms of the form $s_n\colon M^{(n)} \to M$ to the category of quasi-isomorphisms $\widetilde{M} \to M$ is cofinal in the sense of [KS06, Definition 2.5.1]. Then the assertion follows from [KS06, Proposition 2.5.2]. □

In [BVK16, C.5.3], Barbieri-Viale and Kahn showed that ${}^t\mathsf{M}_1(k)[1/p]$ is an abelian category if k is a perfect field. Using Proposition 2.2.11, we can check that their proof also works over arbitrary fields. We can also use the following result to extend their theorem from perfect fields to arbitrary fields.

Lemma 4.1.10. *Let k be a field of characteristic $p > 0$ and let K/k be a purely inseparable extension. Then the extension of scalars*

$$\pi^*\colon {}^t\mathsf{M}_1(k)[1/p] \longrightarrow {}^t\mathsf{M}_1(K)[1/p],$$
$$[L \to G] \longmapsto [L_K \to G_K]$$

is an equivalence of categories.

Proof. First, we have to explain that this functor is well-defined. In other words, the extension of scalars of effective 1-motives

$$\pi^*_{\mathrm{eff}}\colon {}^t\mathsf{M}_1^{\mathrm{eff}}(k) \longrightarrow {}^t\mathsf{M}_1^{\mathrm{eff}}(K)$$

sends a quasi-isomorphism to a quasi-isomorphism. Since the extension of scalars of group schemes preserves kernels and cokernels, the functor π^*_{eff} sends a pullback diagram to a pullback diagram.

Next, we show that π^* is fully faithful. Note that $(L_K)^{(n)} = (L^{(n)})_K$, $(M_K)^{(n)} = (M^{(n)})_K$ and $s_n : M_K^{(n)} \to M_K$ is the base change of $s_n : M^{(n)} \to M$. By Proposition 4.1.9, it suffices to show the isomorphisms

$$\mathrm{Hom}_{\mathrm{eff}}(M^{(n)}, M') \simeq \mathrm{Hom}_{\mathrm{eff}}(M_K^{(n)}, M'_K)$$

for any 1-motives with torsion M, M' over k and for any positive integer n prime to p. We can prove it by copying the proof of Theorem 2.3.8 (purely inseparable extensions are primary).

Finally, we show that π^* is essentially surjective. Let $[L \xrightarrow{u} G]$ be a 1-motive with torsion over K. Since K/k is purely inseparable, there exists a constructible group scheme L_0 over k such that $(L_0)_K \simeq L$. By [Bri17, Lemma 3.10], there exists a semi-abelian variety G_0 over k and an epimorphism $f\colon G \to (G_0)_K$ such that $\ker(f)$ is infinitesimal. Using Lemma 3.7.5, we have a p-power q such that qfu is the base change of a k-morphism $v\colon L_0 \to G_0$. Clearly, in ${}^t\mathsf{M}_1(K)[1/p]$, the 1-motives with torsion $[L \xrightarrow{u} G]$ and $[(L_0)_K \xrightarrow{v_K} (G_0)_K]$ are isomorphic to each other. □

Theorem 4.1.11. *The category ${}^t\mathsf{M}_1(k)[1/p]$ is abelian.*

Proof. In [BVK16, C.5.3], Barbieri-Viale and Kahn proved it over perfect fields. For a general field k, the category ${}^t\mathsf{M}_1(k)[1/p]$ is equivalent to ${}^t\mathsf{M}_1(k^{\mathrm{perf}})[1/p]$ by Lemma 4.1.10, where k^{perf} is a perfect closure of k. Since ${}^t\mathsf{M}_1(k^{\mathrm{perf}})[1/p]$ is abelian, so is ${}^t\mathsf{M}_1(k)[1/p]$. □

Proposition 4.1.12 ([BVK16, C.7])**.** (1) *The natural functor*

$$\mathsf{M}_1(k)[1/p] \longrightarrow {}^t\mathsf{M}_1(k)[1/p]$$

is fully faithful and has a left adjoint $M \mapsto M_{\mathrm{fr}}$.

(2) *The natural functor*

$${}^t\mathsf{M}_1^{\mathrm{tor}}(k)[1/p] \longrightarrow {}^t\mathsf{M}_1(k)[1/p]$$

is fully faithful and has a right adjoint $M \mapsto M_{\mathrm{tor}}$.

In the remainder of this subsection, we show that ${}^t\mathsf{M}_1(k)[1/p]$ is Noetherian. This result will not be used later in this paper. We record it here for interested readers and for potential future reference.

Lemma 4.1.13. *Let $M = [L \to G]$ be a 1-motive with torsion. The following are equivalent:*

(1) *$M = 0$ in ${}^t\mathsf{M}_1(k)[1/p]$;*

(2) *$G = 0$ in $\mathsf{Grp}(k)$, and $\#L(k_s)$ is a power of p.*

Proof. Note that in an abelian category, $X = 0$ if and only if $\mathrm{id}_X = 0$. Thus $M = 0$ in ${}^t\mathsf{M}_1(k)[1/p]$ if and only if $\mathrm{id}_M = 0$, i.e., $p^n\,\mathrm{id}_M = 0$ in ${}^t\mathsf{M}_1(k)$ for some $n \in \mathbb{N}$. This means that $p^n L = 0$ and $p^n G = 0$ as group schemes. Equivalently, $G = 0$ and $\#L(k_s)$ is a power of p. □

Lemma 4.1.14. *Let L be a constructible group scheme over k.*

(1) *Every monomorphism to $[L \to 0]$ in ${}^t\mathsf{M}_1(k)[1/p]$ can be represented by a monomorphism $f : L' \to L$ in ${}^t\mathsf{M}_0(k)$.*
(2) *The 1-motive $[L \to 0]$ is Noetherian in ${}^t\mathsf{M}_1(k)[1/p]$.*

Proof. (1) Every monomorphism to $[L \to 0]$ in ${}^t\mathsf{M}_1(k)[1/p]$ can be represented by an effective morphism

$$(f,0) : [L' \to G'] \to [L \to 0].$$

Thus $\ker(f,0) = [\ker(f) \to G']$ is zero in ${}^t\mathsf{M}_1(k)[1/p]$. By Lemma 4.1.13, $G' = 0$ and $\#\ker(f)(k_s)$ is a power of p. Thus the canonical morphism $[L' \to 0] \to [L'/\ker(f) \to 0]$ is an isomorphism in ${}^t\mathsf{M}_1(k)[1/p]$ and $(f,0) : [L' \to 0] \to [L \to 0]$ factors through it as effective morphisms. It means that $(f,0)$ can be represented by the induced morphism

$$[L'/\ker(f) \to 0] \to [L \to 0].$$

(2) Let $M_1 \hookrightarrow M_2 \hookrightarrow \cdots$ be an ascending chain of subobjects of $[L \to 0]$ in ${}^t\mathsf{M}_1(k)[1/p]$. By (1), each morphism $M_n \hookrightarrow [L \to 0]$ can be represented by an effective morphism

$$(f_n, 0) : [L_n \to 0] \hookrightarrow [L \to 0]$$

with $f_n : L_n \to L$ a monomorphism in ${}^t\mathsf{M}_0(k)$. Since there are no nontrivial quasi-isomorphisms to $[L_n \to 0]$, the monomorphism $[L_n \to 0] \to [L_{n+1} \to 0]$ is of the form

$$\left(\frac{i_n}{p^{\alpha_n}}, 0\right) : [L_n \to 0] \to [L_{n+1} \to 0],$$

where $i_n : L_n \to L_{n+1}$ is a morphism in ${}^t\mathsf{M}_0(k)$. Because $f_{n+1} \circ i_n = p^{\alpha_n} f_n$ is a monomorphism in ${}^t\mathsf{M}_0(k)$, the morphism i_n is also a monomorphism in ${}^t\mathsf{M}_0(k)$. Thus

$$\operatorname{rank} L_1(k_s) \le \operatorname{rank} L_2(k_s) \le \cdots \le \operatorname{rank} L(k_s).$$

So there exists a positive integer N such that

$$\operatorname{rank} L_n(k_s) = \operatorname{rank} L_N(k_s) \quad \text{and} \quad \#L_n(k_s)/L_N(k_s) < +\infty, \quad \forall\, n \ge N.$$

We get an ascending chain

$$[L_{N+1}/L_N \to 0] \hookrightarrow [L_{N+2}/L_N \to 0] \hookrightarrow [L_{N+3}/L_N \to 0] \hookrightarrow \cdots$$

of subobjects of $[L/L_N \to 0]$. Thus

$$\#(L_{N+1}/L_N)(k_s) \le \#(L_{N+2}/L_N)(k_s) \le \cdots \le \#(L/L_N)_{\mathrm{tor}}(k_s).$$

It follows that there exists an integer $N' \ge N$ such that

$$\#(L_n/L_N)(k_s) = \#(L_{N'}/L_N)(k_s), \ \forall\, n \ge N'.$$

It means that the monomorphisms $[L_n/L_N \to 0] \hookrightarrow [L_{n+1}/L_N \to 0]$ are isomorphisms for $n \ge N'$. So the morphisms $[L_n \to 0] \to [L_{n+1} \to 0]$ are isomorphisms for $n \ge N'$. Hence $[L \to 0]$ is Noetherian in ${}^t\mathsf{M}_1(k)[1/p]$. □

Lemma 4.1.15. *Let G be a semi-abelian variety over k.*

(1) *Every monomorphism to* $[0 \to G]$ *in* ${}^t\mathsf{M}_1(k)[1/p]$ *can be represented by a morphism* $g : G' \to G$ *of semi-abelian varieties with* $\ker(g)$ *finite over* k.
(2) *The* 1*-motive* $[0 \to G]$ *is Noetherian in* ${}^t\mathsf{M}_1(k)[1/p]$.

Proof. (1) Every monomorphism to $[0 \to G]$ in ${}^t\mathsf{M}_1(k)[1/p]$ can be represented by an effective morphism

$$(0, g) : [L' \to G'] \to [0 \to G].$$

Thus $\ker(0, g)$ is zero in ${}^t\mathsf{M}_1(k)[1/p]$. Recall that $\ker(0, g) = [L'_0 \to \ker(g)^0_{\mathrm{red}}]$, where L'_0 is the pullback of L' along the closed immersion $\ker(g)^0_{\mathrm{red}} \to \ker(g)$. By Lemma 4.1.13, $\ker(g)^0_{\mathrm{red}} = 0$ and $\#L'_0(k_s)$ is a power of p. Thus the canonical morphism $[L' \to G'] \to [L'/L'_0 \to G']$ is an isomorphism in ${}^t\mathsf{M}_1(k)[1/p]$ and the morphism $(0, g) : [L' \to G'] \to [0 \to G]$ factors through it as effective morphisms. Since $\ker(g)^0_{\mathrm{red}} = 0$, the group scheme L'_0 is the kernel of the induced morphism $u : L' \to \ker(g)$ and the group scheme $\ker(g)$ is finite over k. Thus $L'/L'_0 = L'/\ker(u)$ is a subgroup of $\ker(g)$, which implies that L'/L'_0 is finite étale over k. So the canonical morphism $[L'/L'_0 \to G'] \to [0 \to G'/(L'/L'_0)]$ is a quasi-isomorphism and the morphism $[L'/L'_0 \to G'] \to [0 \to G]$ factors through it. In conclusion, in ${}^t\mathsf{M}_1(k)[1/p]$, the morphism $(0, g)$ can be represented by the induced morphism

$$[0 \to G'/(L'/L'_0)] \longrightarrow [0 \to G],$$

which satisfies the desired properties.

(2) Let $M_1 \hookrightarrow M_2 \hookrightarrow \cdots$ be an ascending chain of subobjects of $[0 \to G]$ in ${}^t\mathsf{M}_1(k)[1/p]$. By (1), each morphism $M_n \hookrightarrow [0 \to G]$ can be represented by an effective morphism

$$(0, g_n) : [0 \to G_n] \hookrightarrow [0 \to G]$$

with $\ker(g_n)$ finite over k. Since all morphisms between Deligne 1-motives are in fact effective, the monomorphism $[0 \to G_n] \to [0 \to G_{n+1}]$ is of the form

$$\left(0, \frac{i_n}{p^{\alpha_n}}\right) : [0 \to G_n] \to [0 \to G_{n+1}],$$

where $i_n : G_n \to G_{n+1}$ is a morphism in $\mathsf{SAV}(k)$. Because p^{α_n} is an isomorphism in ${}^t\mathsf{M}_1(k)[1/p]$, the morphism $(0, i_n) : [0 \to G_n] \to [0 \to G_{n+1}]$ is also a monomorphism in ${}^t\mathsf{M}_1(k)[1/p]$, which implies that $\ker(i_n)$ is a finite group scheme over k. Thus

$$\dim G_1 \le \dim G_2 \le \cdots \le \dim G.$$

So there exists a positive integer N such that

$$\dim G_n = \dim G_N, \ \forall\ n \ge N.$$

We get an ascending chain of subobjects of $\ker(g_N)$ in $\mathsf{Grp}(k)[1/p]$:

$$\ker(i_N) \hookrightarrow \ker(i_{N+1} \circ i_N) \hookrightarrow \ker(i_{N+2} \circ i_{N+1} \circ i_N) \hookrightarrow \cdots$$

Taking Cartier duality of finite (flat) group schemes over k, we get a chain of quotients of $\ker(g_N)^\vee$:

$$\begin{tikzcd}
 & & \ker(g_N)^\vee \arrow[dl, "q_{N+2}"'] \arrow[d, "q_{N+1}"] \arrow[dr, "q_N"] & \\
\cdots \arrow[r, two heads] & \ker(i_{N+2} \circ i_{N+1} \circ i_N)^\vee \arrow[r, two heads] & \ker(i_{N+1} \circ i_N)^\vee \arrow[r, two heads] & \ker(i_N)^\vee.
\end{tikzcd}$$

Since $\ker(g_N)^\vee$ is a Noetherian scheme, the descending chain of its subobjects

$$\ker(q_N) \supseteq \ker(q_{N+1}) \supseteq \ker(q_{N+2}) \supseteq \cdots$$

is stationary. Thus there exists an integer $N' \geq N$ such that

$$\ker(i_n \circ \cdots \circ i_N) = \ker(i_{n+1} \circ \cdots \circ i_N), \ \forall\ n \geq N',$$

It follows that $i_n : G_n \to G_{n+1}$ are isomorphisms in $\mathsf{Grp}(k)[1/p]$ for $n \geq N'$. Hence $[0 \to G]$ is Noetherian in ${}^t\mathsf{M}_1(k)[1/p]$. □

Theorem 4.1.16. *Let k be a field. Then the category ${}^t\mathsf{M}_1(k)[1/p]$ is Noetherian.*

Proof. Let $[L \to G]$ be a 1-motive with torsion over k. Consider the canonical short exact sequence

$$0 \to [0 \to G] \to [L \to G] \to [L \to 0] \to 0.$$

By Lemma 4.1.14 and 4.1.15, the 1-motives $[L \to 0]$ and $[0 \to G]$ are Noetherian in ${}^t\mathsf{M}_1(k)[1/p]$. Thus the 1-motive $[L \to G]$ is Noetherian in ${}^t\mathsf{M}_1(k)[1/p]$. Hence the category ${}^t\mathsf{M}_1(k)[1/p]$ is Noetherian. □

4.2. **Perverting t-structures.** Recall that a t-structure on a triangulated category $\mathcal{D}$ is a pair of full subcategories satisfying three simple axioms ([BBD82, Définition 1.3.1]). Let $\mathcal{D}$ be a triangulated category endowed with a t-structure $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$. We denote by $\mathcal{D}^\heartsuit$ its heart $\mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$. For $n \in \mathbb{Z}$, we denote by $\tau^{\leq n}$ and $\tau^{\geq n}$ the truncation functors with respect to this t-structure. We also write $H^n X = \tau^{\leq n}\tau^{\geq n} X[n]$, which is an object of $\mathcal{D}^\heartsuit$.

We consider full subcategories of $\mathcal{D}^\heartsuit$ with the following properties.

Hypothesis 4.2.1 ([Ayo11, Hypothesis 2.1])**.** (1) $\mathcal{A}$ is a Serre subcategory of $\mathcal{D}^\heartsuit$, i.e., stable under subobjects, quotients and extensions.

(2) The inclusion $\mathcal{A} \hookrightarrow \mathcal{D}^\heartsuit$ admits a left adjoint $F\colon \mathcal{D}^\heartsuit \to \mathcal{A}$.

(3) If $0 \to X' \to X \to X'' \to 0$ is an exact sequence in $\mathcal{D}^\heartsuit$ with $X'' \in \mathcal{A}$, then $F(X') \to F(X)$ is a monomorphism.

Definition 4.2.2 ([Ayo11, Definition 2.3])**.** An object $X \in \mathcal{D}^\heartsuit$ is said to be F-connected if $F(X) = 0$.

Remark 4.2.3. There is no nontrivial morphisms from F-connected objects to objects in $\mathcal{A}$. In fact, if X' is F-connected and X'' is in $\mathcal{A}$, then

$$\mathrm{Hom}_{\mathcal{D}^\heartsuit}(X', X'') \simeq \mathrm{Hom}_{\mathcal{A}}(FX', X'') = 0.$$

As a result, if $0 \to X' \to X \to X'' \to 0$ is an exact sequence in $\mathcal{D}^\heartsuit$ where X' is F-connected and $X'' \in \mathcal{A}$, then $X'' \simeq FX$.

Definition 4.2.4. We define two full subcategories of $\mathcal{D}$ as follows:

- ${}^t\mathcal{D}^{\leq 0}$ is the full subcategory consisting of $P \in \mathcal{T}$ such that $H^i(P) = 0$ for $i > 1$ and $H^1(P)$ is F-connected.
- ${}^t\mathcal{D}^{\geq 0}$ is the full subcategory consisting of $N \in \mathcal{T}$ such that $H^i(N) = 0$ for $i < 0$ and $H^0(N) \in \mathcal{A}$.

Proposition 4.2.5 ([Ayo11, Proposition 2.4])**.** *The pair $({}^t\mathcal{D}^{\leq 0}, {}^t\mathcal{D}^{\geq 0})$ is a t-structure on $\mathcal{D}$.*

Definition 4.2.6. The t-structure $({}^t\mathcal{D}^{\le 0}, {}^t\mathcal{D}^{\ge 0})$ defined above is called the $\mathcal{A}$-perverted t-structure. We denote by ${}^t\mathcal{D}^{\heartsuit}$ its heart ${}^t\mathcal{D}^{\le 0} \cap {}^t\mathcal{D}^{\ge 0}$. For $n \in \mathbb{Z}$, we denote by ${}^t\tau^{\le n}$ and ${}^t\tau^{\ge n}$ the truncation functors with respect to this t-structure. We also write the cohomology with respect to this t-structure as ${}^tH^n$.

Remark 4.2.7 ([Ayo11, Remark 2.6])**.** By definition, an object X of $\mathcal{D}$ is in ${}^t\mathcal{D}^{\heartsuit}$ if and only if it satisfies the following properties:

(1) $H^i(X) = 0$ for $i \notin \{0,1\}$;
(2) $H^0(X) \in \mathcal{A}$;
(3) $H^1(X)$ is F-connected.

Remark 4.2.8. By definition, if the old t-structure on $\mathcal{D}$ is non-degenerate, i.e.,

$$\bigcap_{n\in\mathbb{Z}} \mathcal{D}^{\le n} = \bigcap_{n\in\mathbb{Z}} \mathcal{D}^{\ge n} = 0,$$

then so is the $\mathcal{A}$-perverted t-structure.

The following result is a generalization of [BVK16, Proposition 3.11.2].

Proposition 4.2.9. *Keep the above notations and assumptions. Let X be an object of $\mathcal{D}$. For any $n \in \mathbb{Z}$,*

(1) $H^m({}^t\tau^{\le n-1}X) = 0$ *for* $m \ge n+1$*; and* $H^m({}^t\tau^{\ge n}X) = 0$ *for* $m \le n-1$*;*
(2) *for* $m \ge n+1$*,*
$$H^m(X) \simeq H^m({}^t\tau^{\ge n}X);$$
(3) *for* $m \le n-1$*,*
$$H^m({}^t\tau^{\le n-1}X) \simeq H^m(X);$$
(4) *we have*
$$H^0({}^tH^nX) \simeq H^n({}^t\tau^{\ge n}X), \quad H^n({}^t\tau^{\le n-1}X) \simeq H^1({}^tH^{n-1}X).$$
and a short exact sequence in $\mathcal{D}^{\heartsuit}$
$$0 \to H^1({}^tH^nX) \to H^{n+1}(X) \to H^0({}^tH^{n+1}X) \to 0;$$
(5) $H^0({}^tH^{n+1}X) \simeq F(H^{n+1}(X))$.

Proof. The first assertion is clear by definition.

Consider the following distinguished triangle in $\mathcal{D}$

$${}^t\tau^{\le n-1}X \longrightarrow X \longrightarrow {}^t\tau^{\ge n}X \longrightarrow {}^t\tau^{\le n-1}X[1].$$

It induces a long exact sequence in $\mathcal{D}^{\heartsuit}$

$$\cdots \to H^m({}^t\tau^{\le n-1}X) \to H^m(X) \to H^m({}^t\tau^{\ge n}X) \to H^{m+1}({}^t\tau^{\le n-1}X) \to \cdots.$$

Combining it with (1), we get (2) and (3).

Now, consider the following distinguished triangle in $\mathcal{D}$

$${}^tH^n(X)[-n] \longrightarrow {}^t\tau^{\ge n}X \longrightarrow {}^t\tau^{\ge n+1}X \longrightarrow {}^tH^n(X)[-n+1].$$

It induces the following exact sequence in $\mathcal{D}^{\heartsuit}$

$$\cdots \to H^{n-1}({}^t\tau^{\ge n+1}X) \to H^n({}^tH^n(X)[-n]) \to H^n({}^t\tau^{\ge n}X) \to H^n({}^t\tau^{\ge n+1}X)$$
$$\to H^{n+1}({}^tH^nX[-n]) \to H^{n+1}({}^t\tau^{\ge n}X) \to H^{n+1}({}^t\tau^{\ge n+1}X) \to H^{n+2}({}^tH^nX[-n]) \to \cdots.$$

By (1), $H^{n-1}({}^t\tau^{\geq n+1}X)$ and $H^n({}^t\tau^{\geq n+1}X)$ both vanish. Note also that

$$H^{n+2}({}^tH^nX[-n]) = 0.$$

Thus we get the isomorphism

$$H^0({}^tH^nX) \simeq H^n({}^t\tau^{\geq n}X)$$

and the exact sequence in (4).

Using the above argument for the distinguished triangle

$${}^t\tau^{\leq n-2}X \longrightarrow {}^t\tau^{\leq n-1}X \longrightarrow {}^tH^{n-1}X[-n+1] \longrightarrow {}^t\tau^{\leq n-2}X[1],$$

we get $H^n({}^t\tau^{\leq n-1}X) \simeq H^1({}^tH^{n-1}X)$.

The last assertion follows from Remark 4.2.3. □

4.3. **1-motives.** We consider the unbounded derived category $D(\mathsf{HI}_{\leq 1}(k,\Lambda))$. The standard t-structure on it is called the homotopy t-structure. The homotopy t-structure on $\mathsf{DM}^{\mathrm{eff}}_{\text{ét}}(k,\Lambda)$ with heart $\mathsf{HI}_{\text{ét}}(k,\Lambda)$ restricts to the above t-structure on $D(\mathsf{HI}_{\leq 1}(k,\Lambda))$, which justifies the name.

The following result is proved in [Ayo11] for $\mathbb{Q}$-coefficients, and we refine it to integral coefficients here (at least inverting the exponential characteristics).

Proposition 4.3.1. *The subcategory $\mathsf{HI}_{\leq 0}(k,\Lambda) \hookrightarrow \mathsf{HI}_{\leq 1}(k,\Lambda)$ satisfies Hypothesis 4.2.1. More precisely,*

(1) *$\mathsf{HI}_{\leq 0}(k,\Lambda)$ is a Serre subcategory of $\mathsf{HI}_{\leq 1}(k,\Lambda)$;*
(2) *the inclusion $\mathsf{HI}_{\leq 0}(k,\Lambda) \hookrightarrow \mathsf{HI}_{\leq 1}(k,\Lambda)$ admits a left adjoint*

$$\pi_0 \colon \mathsf{HI}_{\leq 1}(k,\Lambda) \to \mathsf{HI}_{\leq 0}(k,\Lambda);$$

(3) *if*

$$0 \longrightarrow \mathscr{F}' \xrightarrow{i} \mathscr{F} \xrightarrow{q} \mathscr{F}'' \longrightarrow 0$$

is an exact sequence in $\mathsf{HI}_{\leq 1}(k,\Lambda)$ with $\mathscr{F}'' \in \mathsf{HI}_{\leq 0}(k,\Lambda)$, then $\pi_0(i)\colon \pi_0(\mathscr{F}') \to \pi_0(\mathscr{F})$ is a monomorphism.

Proof. The first assertion has been proved in Corollary 3.6.10.

Clearly, the restriction of π_0 to $\mathsf{HI}_{\leq 1}(k,\Lambda)$ is left adjoint to $\mathsf{HI}_{\leq 0}(k,\Lambda) \hookrightarrow \mathsf{HI}_{\leq 1}(k,\Lambda)$.

Now, we prove the last assertion. By Lemma 3.8.6, the composition $\mathscr{F}^0 \to \mathscr{F} \to \mathscr{F}''$ is trivial. Thus the morphism $\mathscr{F}^0 \to \mathscr{F}$ factors through $\mathscr{F}'$. Again, the induced morphism $\mathscr{F}^0 \to \pi_0(\mathscr{F}')$ is trivial and thus the morphism $\mathscr{F}^0 \to \mathscr{F}'$ factors through $\mathscr{F}'^0$.

$$\begin{tikzcd}
 & \mathscr{F}'^0 \arrow[r, shift left, "f"] \arrow[d] & \mathscr{F}^0 \arrow[l, shift left, "g"] \arrow[d] \arrow[dl] & & \\
0 \arrow[r] & \mathscr{F}' \arrow[r] \arrow[d] & \mathscr{F} \arrow[r] \arrow[d] & \mathscr{F}'' \arrow[r] & 0 \\
 & \pi_0(\mathscr{F}') \arrow[r] & \pi_0(\mathscr{F}) & &
\end{tikzcd}$$

It is easy to check that f and g are inverses of each other. Applying the snake lemma to the top two rows of the above diagram (with zero being the third term of the first row), we get that $\pi_0(\mathscr{F}') \to \pi_0(\mathscr{F})$ is a monomorphism. □

We apply the construction from the previous subsection.

Definition 4.3.2. The 1-motivic t-structure on $D(\mathsf{HI}_{\leq 1}(k,\Lambda))$ is the $\mathsf{HI}_{\leq 0}(k,\Lambda)$-perverted t-structure associated with the homotopy t-structure. The heart of this 1-motivic t-structure will be denoted by $\mathsf{MM}_1(k,\Lambda)$. We call objects in $\mathsf{MM}_1(k,\Lambda)$ 1-motives.

As explained in Remark 4.2.7, an object X in $D(\mathsf{HI}_{\leq 1}(k,\Lambda))$ is a 1-motive if and only if it satisfies the following properties:

(1) $H^i(X) = 0$ for $i \notin \{0,1\}$;
(2) $H^0(X)$ is a 0-motivic sheaf;
(3) $H^1(X)$ is a connected 1-motivic sheaf.

By truncation, in fact, we may and do represent X by a two-term complex concentrated in degrees $0,1$. We write it as $[L \to G]$ with Deligne 1-motives as main examples. We call it a 0-motive if it is quasi-isomorphic to $[L \to 0]$ with L a 0-motivic sheaf. We will call it constructible if $H^0(X)$ and $H^1(X)$ are finitely presented 1-motivic sheaves in the sense of Definition 3.8.1.

There is a functor
$$T\colon {}^t\mathsf{M}_1(k)[1/p] \to \mathsf{MM}_1(k)$$
which sends a 1-motive with torsion $[L \to G]$ to the 1-motive $[L_\Lambda^{\mathrm{tr}} \to G_\Lambda^{\mathrm{tr}}]$.

Proposition 4.3.3. *The functor T induces an exact full embedding of ${}^t\mathsf{M}_1(k)[1/p]$ into $\mathsf{MM}_1(k)$. Moreover, with $\mathbb{Q}$-coefficients, the category of 1-motives with torsion is equivalent to the category of constructible 1-motives.*

Proof. This is a refinement of [Ayo11, Proposition 3.11] from rational coefficients to $\mathbb{Z}[1/p]$-coefficients. By dévissage, we have to check the isomorphisms between Homs and Extensions of 1-motives of the form $[L \to 0]$ and $[0 \to G]$. Such isomorphisms can be found in [BVK16, C.8]. With $\mathbb{Q}$-coefficients, Ayoub and Barbieri-Viale [ABV09, Proposition 2.4.10] showed that the cohomological dimension of $\mathsf{HI}_{\leq 1}$ is 1, and then Ayoub [Ayo11, Lemma 3.10] used it to show that a 1-motive M decomposes into a direct sum
$$M \simeq H^0M[0] \oplus H^1M[-1].$$
When M is a constructible 1-motive, this direct sum is a 1-motive with torsion. □

4.4. **Higher direct images of 1-motives.** Let K/k be a field extension. By abuse of notation, we denote the direct image functor of 1-motivic sheaves as e_*, which is the functor e_{1*} in Definition 3.4.6. For a complex $X \in D(\mathsf{HI}_{\leq 1}(K,\Lambda))$, we denote Re_*X the total derived direct image of X, which is a complex of 1-motivic sheaves over k. We denote by R^ie_*X (resp. ${}^mR^ie_*X = [L^i \to G^i]$) the i-th cohomology of Re_*X relative to the homotopy (resp. 1-motivic) t-structure on $D(\mathsf{HI}_{\leq 1}(k,\Lambda))$.

Theorem 4.4.1. *Let K/k be a field extension and let L be a 0-motivic sheaf over K. Then for all $i \geq 0$,*
$${}^mR^ie_*[L \to 0] = [R^ie_*L \to 0].$$
In particular, ${}^mR^ie_*[L \to 0]$ is a torsion 0-motive for all $i \geq 1$. Moreover, if K/k is primary, then
$${}^mR^ie_*[L \to 0] = [H^i(\Gamma, L(K_s)) \to 0]$$
where $\Gamma = \mathrm{Gal}(K_s/Kk_s)$ and $H^i(\Gamma, L(K_s))$ is the 0-motivic sheaf associated with the $\mathrm{Gal}(k_s/k)$-module $H^i(\Gamma, L(K_s))$.

Proof. By Corollary 3.6.14 (1), $R^i e_* L$ is a 0-motivic sheaf for any $i \geq 0$ and is torsion for any $i \geq 1$. Applying Proposition 4.2.9 (5) to $Re_* L$, we get for any $i \geq 0$,

$$\ker(L^i \to G^i) \simeq \pi_0(R^i e_* L) \simeq R^i e_* L$$

and

$$\operatorname{coker}(L^i \to G^i) \simeq (R^{i+1} e_* L)^0 = 0.$$

This means that $[L^i \to G^i]$ is quasi-isomorphic to $[R^i e_* L \to 0]$. □

When K/k is primary, we can use results on cohomology of profinite groups to deduce some properties of the higher direct images of 0-motivic sheaves.

Lemma 4.4.2. *Let Γ be a profinite group and L be a discrete Γ-module which is a finitely generated free abelian group. Then $H^1(\Gamma, L)$ is finite.*

Proof. This result is well-known to experts. For example, it is [Har20, Exercise 4.10]. For readers' convenience, we give a proof. By definition of discrete Γ-modules, for any $x \in L$, the stabilizer $\Gamma_x = \{g \in \Gamma \mid gx = x\}$ is open in Γ. Since L is finitely generated, the group

$$U := \{g \in \Gamma \mid gx = x, \text{ for all } x \in L\}$$

is an open normal subgroup of Γ. The Hochschild–Serre spectral sequence [Ser02, Chapter I, 2.6 b)] gives us the exact sequence

$$0 \to H^1(\Gamma/U, L^U) \to H^1(\Gamma, L) \to H^1(U, L)^{\Gamma/U}.$$

We claim that $H^1(U, L)$ vanishes. Then it suffices to show the finiteness of $H^1(\Gamma/U, L^U)$. It is finitely generated because L^U is finitely generated. Then it is finite as the higher cohomology groups are torsion groups.

Now, we prove the claim. By [Ser02, Chapter I, 2.2, Corollary 1], we have

$$H^1(U, L) \simeq \varinjlim H^1(U/V, L^V),$$

where V runs over all open normal subgroups of U. Thus we reduce to show $H^1(U/V, L^V) = 0$. By construction, the group U acts trivially on L. So U/V acts trivially on $L^V = L$. It follows that $H^1(U/V, L^V)$ is the group of homomorphisms from U/V to L. Since U/V is finite and L is free, this group vanishes. □

Remark 4.4.3. It is necessary to assume that L is free in Lemma 4.4.2. A counterexample is that $\Gamma = \mathrm{Gal}(k_s/k)$ and $L = \mu_n(k_s) = \{x \in k_s \mid x^n = 1\}$ for n prime to $\mathrm{char}(k)$. Taking the long exact sequence of cohomology associated with the Kummer exact sequence

$$0 \to \mu_n(k_s) \to k_s^\times \to k_s^\times \to 0,$$

and applying Hilbert's theorem 90, we obtain $H^1(\mathrm{Gal}(k_s/k), \mu_n(k_s)) \simeq k^\times / k^{\times n}$. It is not finite in general.

So we have the following result.

Lemma 4.4.4. *Let K/k be a primary field extension and let L be a lattice over K. Then ${}^m R^1 e_* [L \to 0]$ is a constructible 0-motive.*

Proof. This is a direct consequence of Theorems 4.4.1 and Lemma 4.4.2. □

Lemma 4.4.5. *Let K/k be a field extension and let G be a connected 1-motivic sheaf over K. Then*

$$^mR^i e_*[0 \to G] = \begin{cases} [0 \to (e_*G)^0], & \text{if } i = 0; \\ [\pi_0(R^1e_*G) \to 0], & \text{if } i = 2; \\ [R^{i-1}e_*G \to 0], & \text{if } i \geq 3 \end{cases}$$

and we have an exact sequence

$$0 \to \pi_0(e_*G) \to L^1 \to G^1 \to (R^1e_*G)^0 \to 0.$$

In particular, $^mR^i e_*[0 \to G]$ are 0-motives for $i \geq 2$ and are torsion for $i \geq 3$.

Proof. By Theorem 3.6.14, R^ie_*G are torsion 0-motivic sheaves for all $i \geq 2$. Applying Proposition 4.2.9 (5) to $Re_*[0 \to G]$, we get for all $i \geq 3$,

$$\ker(L^i \to G^i) \simeq \pi_0(R^ie_*[0 \to G]) \simeq R^{i-1}e_*G$$

and

$$\operatorname{coker}(L^{i-1} \to G^{i-1}) \simeq (R^ie_*[0 \to G])^0 = 0.$$

This means that $[L^i \to G^i]$ is quasi-isomorphic to $[R^{i-1}e_*G \to 0]$ for $i \geq 3$, and $\operatorname{coker}(L^2 \to G^2) = 0$. Note that

$$\ker(L^2 \to G^2) \simeq \pi_0(R^2e_*[0 \to G]) \simeq \pi_0(R^1e_*G).$$

Thus $[L^2 \to G^2]$ is quasi-isomorphic to $[\pi_0(R^1e_*G) \to 0]$. We also have

$$\ker(L^0 \to G^0) \simeq \pi_0(R^{-1}e_*G) = 0,$$

and

$$\operatorname{coker}(L^0 \to G^0) \simeq (e_*G)^0.$$

Thus $[L^0 \to G^0]$ is quasi-isomorphic to $[0 \to (e_*G)^0]$. □

Theorem 4.4.6. *Let K/k be a primary field extension and let A be an abelian variety over K. Then*

$$^mR^i e_*[0 \to A] = \begin{cases} [0 \to \pi_*A], & \text{if } i = 0; \\ [\mathrm{LN}(A, Kk_s/k_s) \to 0], & \text{if } i = 1; \\ [R^{i-1}e_*A \to 0], & \text{if } i \geq 2. \end{cases}$$

In particular, $^mR^0e_*[0 \to A]$ is a constructible 1-motive, and $^mR^ie_*[0 \to A]$ are torsion 0-motives for $i \geq 2$. Moreover, if K/k is a finitely generated regular extension, then $^mR^1e_*[0 \to A]$ is a constructible 0-motive.

Proof. By Lemma 4.4.5, it suffices to compute $^mR^0e_*$ and $^mR^1e_*$. This involves the information of e_*A and R^1e_*A. By Theorem 3.8.7,

$$(e_*A)^0 = \pi_*A \quad \text{and} \quad \pi_0(e_*A) = \mathrm{LN}(A, Kk_s/k_s).$$

Thus $^mR^0e_*[0 \to A] = [0 \to \pi_*A]$ and

$$\ker(L^1 \to G^1) \simeq \pi_0(R^1e_*[0 \to A]) \simeq \pi_0(e_*A) \simeq \mathrm{LN}(A, Kk_s/k_s).$$

By Theorem 3.8.10, R^1e_*A is a torsion 0-motivic sheaf. So

$$\operatorname{coker}(L^1 \to G^1) = (R^1e_*A)^0 = 0.$$

Hence $[L^1 \to G^1]$ is quasi-isomorphic to $[\mathrm{LN}(A, Kk_s/k_s) \to 0]$. □

Theorem 4.4.7. *Let X be a smooth projective and geometrically connected variety over k and let K be the function field of X. Then we have*

$$
{}^m R^i e_*[0 \to \mathbb{G}_m] = \begin{cases} [0 \to \mathbb{G}_m], & \text{if } i = 0; \\ 0, & \text{if } i = 2; \\ [R^{i-1} e_* \mathbb{G}_m \to 0], & \text{if } i \geq 3, \end{cases}
$$

and an exact sequence

$$
0 \to (Kk_s)^\times / k_s^\times \to L^1 \to G^1 \to R^1 e_* \mathbb{G}_m \to 0,
$$

where $R^1 e_* \mathbb{G}_m$ is the cokernel of $\mathrm{Div}^0(X_{k_s}) \to \mathrm{Pic}^0_{X/k}$.

Moreover, with $\mathbb{Q}$-coefficients, we have

$$
{}^m R^1 e_*[0 \to \mathbb{G}_m] = [\mathrm{Div}^0(X_{k_s}) \to \mathrm{Pic}^0_{X/k}].
$$

Proof. By Lemma 4.4.5, it suffices to compute ${}^m R^i e_*$ for $i = 0, 1, 2$. By Theorem 3.8.11, $R^1 e_* \mathbb{G}_m$ is a connected 1-motivic sheaf. Thus

$$
{}^m R^2 e_*[0 \to \mathbb{G}_m] = 0, \quad \text{and} \quad \operatorname{coker}(L^1 \to G^1) = R^1 e_* \mathbb{G}_m.
$$

By Theorem 3.8.7,

$$
(e_* \mathbb{G}_m)^0 = \pi_* \mathbb{G}_m = \mathbb{G}_m \quad \text{and} \quad \pi_0(e_* \mathbb{G}_m) = (Kk_s)^\times / k_s^\times.
$$

Thus ${}^m R^0 e_*[0 \to \mathbb{G}_m] = [0 \to \mathbb{G}_m]$ and

$$
\ker(L^1 \to G^1) \simeq \pi_0(R^1 e_*[0 \to \mathbb{G}_m]) \simeq \pi_0(e_* \mathbb{G}_m) \simeq (Kk_s)^\times / k_s^\times.
$$

Note that we have the following exact sequence

$$
0 \to k_s^\times \to (Kk_s)^\times \to \mathrm{Div}^0(X_{k_s}) \to \mathrm{Pic}^0_{X/k} \to R^1 e_* \mathbb{G}_m \to 0,
$$

where the exactness at the last three terms is given by Theorem 3.8.11. So the two-term complexes $[L^1 \to G^1]$ and $[\mathrm{Div}^0(X_{k_s}) \to \mathrm{Pic}^0_{X/k}]$ have the same cohomology. If we work with $\mathbb{Q}$-coefficients, then these two complexes are equal in the derived category of 1-motivic sheaves because the cohomological dimension of $\mathsf{HI}_{\leq 1}$ is 1 by [ABV09, Proposition 2.4.10]. □

We can also describe the higher direct images of more general Deligne 1-motives.

Lemma 4.4.8. *Let K/k be a field extension let $M = [L \to G]$ be a Deligne 1-motive over K.*

1. *$R^i e_* M$ are 0-motivic sheaves for $i = 0$ and $i \geq 3$, and they are torsion for $i \geq 3$.*
2. *If G is an abelian variety, then $R^2 e_* M$ is also a 0-motivic sheaves.*
3. *If G is an abelian variety and K/k is a finitely generated regular field extension, then $R^1 e_* M$ is a finitely presented 1-motivic sheaf.*

Proof. The distinguished triangle

$$
G[-1] \to M \to L \to G
$$

induces the following long exact sequence

$$
0 \to R^0 e_* M \to R^0 e_* L \to R^0 e_* G \to R^1 e_* M \to \cdots.
$$

By Theorem 3.6.14, $R^i e_*(L)$ are torsion 0-motivic sheaves for $i \geq 1$ and $R^i e_* G$ are torsion 0-motivic sheaves for $i \geq 2$. By Theorem 3.8.10, $R^1 e_* G$ is also a torsion 0-motivic sheaf if G is an abelian variety. By Corollary 3.8.8 and Lemma 4.4.4, if G is an abelian variety and K/k

is a finitely generated regular field extension, then e_*G and R^1e_*L are finitely presented 1-motivic sheaves. Recall from Corollary 3.6.10 that $\mathsf{HI}_{\leq 0}$ is a Serre subcategory of $\mathsf{HI}_{\leq 1}$. Then the expected result follows from the above long exact sequence. □

Lemma 4.4.9. *Let K/k be a field extension and let $M = [L \to G]$ be a Deligne 1-motive over K.*

(1) *We have*

$$ {}^{m}R^i e_*[L \to G] = \begin{cases} [\pi_0(R^2e_*M) \to 0], & \text{if } i = 2; \\ [R^ie_*M \to 0], & \text{if } i \geq 3. \end{cases} $$

(2) *If $G = A$ is an abelian variety, then we have*

$$ {}^{m}R^i e_*[L \to A] = \begin{cases} [\pi_0(R^1e_*M) \to 0], & \text{if } i = 1; \\ [R^ie_*M \to 0], & \text{if } i \geq 2. \end{cases} $$

Proof. These are direct consequences of Proposition 4.2.9 and Lemma 4.4.8. □

The following result can be viewed as a generalization of the Lang-Néron theorem for Deligne 1-motives of the form $[L \to A]$ with A an abelian variety.

Theorem 4.4.10. *Let K/k be a finitely generated regular field extension and let $M = [L \to A]$ be a Deligne 1-motive over K where A is an abelian variety. Write $\Gamma = \mathrm{Gal}(K_s/Kk_s)$. Then we have an exact sequence of 0-motivic sheaves*

$$ 0 \to X \to \pi_0(R^1e_*M) \to Y \to 0, $$

where

$$ X = \mathrm{coker}(L(K_s)^\Gamma \to \mathrm{LN}(A, Kk_s/k_s)) $$

and

$$ Y = \ker(H^1(\Gamma, L(K_s)) \to R^1e_*A). $$

In particular,

(1) *$\pi_0(R^1e_*M)(k_s)$ is a finitely generated $\mathrm{Gal}(k_s/k)$-module;*
(2) *${}^{m}R^1e_*M = [\pi_0(R^1e_*M) \to 0]$ is a constructible 0-motive.*

Proof. The distinguished triangle in $D(\mathsf{HI}_{\leq 1}(K, \Lambda))$

$$ G[-1] \to M \to L \to G $$

induces the following exact sequence

$$ \cdots \to e_*L \to e_*G \to R^1e_*M \to R^1e_*L \to R^1e_*G \to \cdots . $$

By Theorem 3.6.13, $e_*L = L(K_s)^\Gamma$ and $R^1e_*L = H^1(\Gamma, L(K_s))$. Taking π_0 of the above exact sequence and using Theorem 3.8.7, we obtain the desired exact sequence. By the Lang-Néron theorem (resp. Lemma 4.4.4), we see that X (resp. Y) is a 0-motivic sheaf associated with a finitely generated (resp. finite) $\mathrm{Gal}(k_s/k)$-module. So the same is true for the extension $\pi_0(R^1e_*M)$. □

Appendix A. Some results on étale cohomology

In this appendix, we prove a smooth base change theorem for non-torsion sheaves and use it to compare the small-étale and smooth-étale topoi.

A.1. **A smooth base change theorem for non-torsion étale sheaves.** Recall the classical smooth base change theorem:

Theorem A.1.1 ([SGA 4_{III}, Exposé XVI, Corollaire 1.2])**.** *Consider the Cartesian diagram of schemes*

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ T \arrow[r, "e"] & S, \end{tikzcd}$$

where f is smooth and e is quasi-compact and quasi-separated. Let Λ be the localization of $\mathbb{Z}$ by inverting the exponential characteristics of all local residue fields of S. If $\mathscr{F}$ is a sheaf of sets (resp. of torsion Λ-modules) on $T_{\text{ét}}$, then the base change morphism

$$\alpha^i_{\mathscr{F}}\colon f^*R^ie_*\mathscr{F} \longrightarrow R^ig_*h^*\mathscr{F}$$

is an isomorphism for $i=0$ (resp. for every i).

In [Den88], Deninger proved a proper base change theorem for non-torsion sheaves. Using the same strategy, we prove the following version of smooth base change theorem.

Theorem A.1.2. *Consider the Cartesian diagram of noetherian schemes*

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ T \arrow[r, "e"] & S, \end{tikzcd}$$

and assume that f is smooth and T is excellent. Let Λ be the localization of $\mathbb{Z}$ by inverting the exponential characteristics of all local residue fields of S. If $\mathscr{F}$ is a sheaf of Λ-modules on $T_{\text{ét}}$, then the base change morphism

$$\alpha^i_{\mathscr{F}}\colon f^*R^ie_*\mathscr{F} \longrightarrow R^ig_*h^*\mathscr{F}$$

is an isomorphism for every i.

The argument in [Den88] works almost word by word in our case, except that we shall use the smooth base change theorem for torsion sheaves (Theorem A.1.1) instead of the proper base change theorem, and that we need some additional conditions on the torsion order of the sheaves, i.e., we use sheaves of Λ-modules rather than abelian sheaves.

For completeness and for readers' convenience, we give a proof of Theorem A.1.2 in this subsection. A key ingredient is the following result.

Lemma A.1.3 ([Den88, 2.2])**.** *Let T be a normal scheme and $e\colon T\to S$ be a morphism of noetherian schemes. Then $R^ie_*(\mathbb{Q})=0$ for $i\geq 1$.*

We use it to prove the smooth base change theorem for the constant sheaves defined by finitely generated Λ-modules.

Lemma A.1.4 (cf. [Den88, 2.3])**.** *Consider the Cartesian diagram of noetherian schemes*

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ T \arrow[r, "e"] & S, \end{tikzcd}$$

ans assume that f is smooth and T is normal. Let Λ be the localization of $\mathbb{Z}$ by inverting the exponential characteristics of all local residue fields of S, and let C be a finitely generated Λ-module. Then the base change morphism

$$\alpha_C^i\colon f^*R^ie_*C \longrightarrow R^ig_*h^*C$$

is an isomorphism for every $i \geq 0$.

Proof. Note that e is quasi-compact and quasi-separated since T is noetherian. In view of the smooth base change theorem for torsion sheaves (Theorem A.1.1), we may assume that $C = \Lambda$. Consider the exact sequence

$$0 \to \Lambda \to \mathbb{Q} \to \mathbb{Q}/\Lambda \to 0.$$

By our assumptions, h is smooth and T is normal. Thus Y is also normal. It follows from Lemma A.1.3 that

$$R^ie_*\mathbb{Q} = 0 \quad \text{and} \quad R^ig_*\mathbb{Q} = 0 \quad \text{for} \quad i \geq 1.$$

Thus the above short exact sequence gives us a commutative diagram with exact rows

$$\begin{tikzcd} 0 \arrow[r] & f^*e_*\Lambda \arrow[r] \arrow[d, "\alpha_\Lambda^0"'] & f^*e_*\mathbb{Q} \arrow[r] \arrow[d, "\alpha_\mathbb{Q}^0"'] & f^*e_*(\mathbb{Q}/\Lambda) \arrow[r] \arrow[d, "\alpha_{\mathbb{Q}/\Lambda}^0"'] & f^*R^1e_*\Lambda \arrow[r] \arrow[d, "\alpha_\Lambda^1"'] & 0 \\ 0 \arrow[r] & g_*h^*\Lambda \arrow[r] & g_*h^*\mathbb{Q} \arrow[r] & g_*h^*(\mathbb{Q}/\Lambda) \arrow[r] & R^1g_*h^*\Lambda \arrow[r] & 0, \end{tikzcd}$$

and a commutative diagram for every $i \geq 2$

$$\begin{tikzcd} f^*R^{i-1}e_*(\mathbb{Q}/\Lambda) \arrow[r] \arrow[d, "\alpha_{\mathbb{Q}/\Lambda}^{i-1}"'] & f^*R^ie_*\Lambda \arrow[d, "\alpha_\Lambda^i"'] \\ R^{i-1}g_*h^*(\mathbb{Q}/\Lambda) \arrow[r] & R^ig_*(h^*\Lambda). \end{tikzcd}$$

By the smooth base change theorem for torsion sheaves (Theorem A.1.1), the $\alpha_{\mathbb{Q}/\Lambda}^i$ are isomorphisms for all i. Hence α_Λ^i is an isomorphism for $i \geq 2$. By Theorem A.1.1 again, for every sheaf of sets $\mathscr{F}$, the base change morphism $f^*e_*\mathscr{F} \to g_*h^*\mathscr{F}$ is an isomorphism. Note that the inverse images and the direct images for sheaves of Λ-modules are compatible with taking the underlying sheaves of sets ([Stacks, Proposition 00YV]). Thus α_Λ^0, $\alpha_\mathbb{Q}^0$ and $\alpha_{\mathbb{Q}/\Lambda}^0$ are isomorphisms. Hence α_Λ^1 is also an isomorphism. □

Lemma A.1.5 (cf. [Den88, 2.4])**.** *Consider the Cartesian diagram of noetherian schemes*

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ T \arrow[r, "e"] & S, \end{tikzcd}$$

ans assume that f is smooth. Let Λ be the localization of $\mathbb{Z}$ by inverting the exponential characteristics of all local residue fields of S, and let $\mathscr{F}$ be a sheaf of Λ-modules on $T_{\text{ét}}$. If $\mathscr{F}$ is of the form $\tau_* C$ where $\tau\colon U \to T$ is a finite morphism with U normal and C is a finitely generated Λ-module, then the base change morphism

$$\alpha^i_{\mathscr{F}}\colon f^* R^i e_* \mathscr{F} \longrightarrow R^i g_* h^* \mathscr{F}$$

is an isomorphism for every $i \geq 0$.

Proof. Consider the following commutative diagram of Cartesian squares

$$\begin{tikzcd} Z \arrow[r, "\tau'"] \arrow[d, "h'"'] & Y \arrow[r, "g"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ U \arrow[r, "\tau"] & T \arrow[r, "e"] & S. \end{tikzcd}$$

By [SGA 4_{III}, Exposé XII, Proposition 4.4(ii)], we have a morphism of spectral sequences

$$\begin{tikzcd} E_2^{p,q} = f^* R^p e_* R^q \tau_* C \arrow[r, Rightarrow] \arrow[d] & f^* R^{p+q} (e\tau)_* C \arrow[d, "\alpha_C^{p+q}"] \\ R^p g_* R^q \tau'_* h'^* C \arrow[r, Rightarrow] & R^{p+q} (g\tau')_* h'^* C. \end{tikzcd}$$

Since τ and τ' are finite morphisms, the direct images τ_* and τ'_* are exact. Thus we have a commutative diagram

$$\begin{tikzcd} f^* R^i e_* (\tau_* C) \arrow[r, "\simeq"] \arrow[d, "\alpha^i_{\tau_* C}"] & f^* R^i (e\tau)_* C \arrow[dd, "\alpha^i_C"] \\ R^i g_* h^* (\tau_* C) \arrow[d, "\alpha^0_C"] & \\ R^i g_* (\tau'_* h'^* C) \arrow[r, "\simeq"] & R^i (g\tau')_* h'^* C. \end{tikzcd}$$

By Lemma A.1.4, the base change morphisms α^0_C and α^i_C in the diagram are isomorphisms. Hence the base change morphism $\alpha^i_{\tau_* C}$ is an isomorphism, i.e., $\alpha^i_{\mathscr{F}}$ is an isomorphism. □

Now we prove the main theorem of this subsection.

Proof of Theorem A.1.2. (cf. [Den88, 2.5]). By [SGA 4_{III}, Exposé IX, Corollaire 2.7.2], every sheaf of Λ-modules $\mathscr{F}$ on $T_{\text{ét}}$ is a filtered colimit of constructible sheaves of Λ-modules. Since e and g are quasi-compact and quasi-separated by our assumptions, the higher direct images $R^i e_*$ and $R^i g_*$ commute with filtered colimits ([SGA 4_{II}, Exposé VII, Corollaire 5.11]). So we may assume that $\mathscr{F}$ is a constructible sheaf of Λ-modules. Because T is excellent, it is a universally Japanese scheme by [EGA IV_2, Scholie 7.8.3(vi)]. Then according to [SGA 4_{III}, Exposé IX, Remarques 2.14.2], there exists a monomorphism

$$\mathscr{F} \hookrightarrow \bigoplus_{i=1}^{n} \tau_{i*} C_i,$$

where C_i is a finitely generated Λ-module and $\tau_i\colon U_i \to T$ is a finite morphism with U_i normal. Denote $\mathscr{G}$ the constructible sheaf of Λ-modules $\bigoplus_{i=1}^n \tau_{i*} C_i$, and denote $\mathscr{H}$ the cokernel of above inclusion $\mathscr{F} \hookrightarrow \mathscr{G}$. Then $\mathscr{H}$ is also a constructible sheaf of Λ-modules. The short exact sequence

$$0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$$

induces the following commutative diagram with exact rows:

$$\begin{tikzcd}
f^* R^i e_* \mathscr{G} \arrow[r] \arrow[d, "\alpha^i_{\mathscr{G}}"', "\simeq"] & f^* R^i e_* \mathscr{H} \arrow[r] \arrow[d, "\alpha^i_{\mathscr{H}}"'] & f^* R^{i+1} e_* \mathscr{F} \arrow[r] \arrow[d, "\alpha^{i+1}_{\mathscr{F}}"'] & f^* R^{i+1} e_* \mathscr{G} \arrow[r] \arrow[d, "\alpha^{i+1}_{\mathscr{G}}"', "\simeq"] & f^* R^{i+1} e_* \mathscr{H} \arrow[d, "\alpha^{i+1}_{\mathscr{H}}"'] \\
R^i g_* h^* \mathscr{G} \arrow[r] & R^i g_* h^* \mathscr{H} \arrow[r] & R^{i+1} g_* h^* \mathscr{F} \arrow[r] & R^{i+1} g_* h^* \mathscr{G} \arrow[r] & R^{i+1} g_* h^* \mathscr{H}.
\end{tikzcd}$$

We prove by induction on i that α^i is an isomorphism for every constructible sheaf of Λ-modules. For $i < 0$, this is trivial. Assume that the assertion holds for a fixed i. Then $\alpha^i_{\mathscr{H}}$ is an isomorphism. By Lemma A.1.5, the base change morphisms $\alpha^i_{\mathscr{G}}$ and $\alpha^{i+1}_{\mathscr{G}}$ are isomorphisms. Thus $\alpha^{i+1}_{\mathscr{F}}$ is a monomorphism. Since $\mathscr{F}$ is an arbitrary constructible sheaf of Λ-modules, $\alpha^{i+1}_{\mathscr{H}}$ is a monomorphism as well. It follows from the five lemma that $\alpha^{i+1}_{\mathscr{F}}$ is an isomorphism, which completes the proof. □

A.2. **Comparing small and smooth étale topoi.** We use the smooth base change theorem for non-torsion sheaves (Theorem A.1.2) to compare the small and smooth étale topoi. The reader may want to compare this subsection with [Stacks, Section 0757].

Let S be a noetherian scheme. Let Sm/S be the category of smooth separated schemes of finite type over S. For $n \in \mathbb{N}$, we denote by $(\mathsf{Sm}/S)_{\leq n}$ the full subcategory of Sm/S whose objects are the smooth schemes over S of relative dimension less than or equal to n. We sometimes write $(\mathsf{Sm}/S)_{\leq 0}$ as Et/S[4].

Let $\mathcal{C}/S$ be the category Sm/S or $(\mathsf{Sm}/S)_{\leq n}$. Let Λ be the localization of $\mathbb{Z}$ by inverting the exponential characteristics of all local residue fields of S. Denote $\mathsf{Shv}_{\text{ét}}(\mathcal{C}/S, \Lambda)$ the category of étale sheaves of Λ-modules on $\mathcal{C}/S$.

For $f\colon X \to S$ in $\mathcal{C}$, the natural inclusion $\sigma_f\colon (\mathsf{Et}/X)_{\text{ét}} \hookrightarrow (\mathcal{C}/S)_{\text{ét}}$ is a continuous functor, i.e., we have a functor[5]

$$\sigma_{f*}\colon \mathsf{Shv}_{\text{ét}}(\mathcal{C}/S, \Lambda) \longrightarrow \mathsf{Shv}_{\text{ét}}(\mathsf{Et}/X, \Lambda),$$
$$\mathscr{F} \longmapsto \mathscr{F} \circ \sigma_f.$$

By [SGA 4_{I}, Exposé III, Proposition 1.2], σ_{f*} admits a left adjoint σ_f^*. We sometimes denote σ_{id_S} by σ_S (or σ if there is no risk of confusion).

Let $e\colon T \to S$ be a morphism of noetherian schemes. Then the base change functor $\mathcal{C}/S \to \mathcal{C}/T$ induces a continuous functor of étale sites[6]. Thus we have a pair of adjunctions

$$e^*_{\mathcal{C}}\colon \mathsf{Shv}_{\text{ét}}(\mathcal{C}/S, \Lambda) \leftrightarrows \mathsf{Shv}_{\text{ét}}(\mathcal{C}/T, \Lambda)\colon e^{\mathcal{C}}_*,$$

where $e^{\mathcal{C}}_* \mathscr{F} = \mathscr{F} \circ e$. When $\mathcal{C} = \mathsf{Et}$, we write $e^*_{\mathcal{C}}$ (resp. $e^{\mathcal{C}}_*$) as e^* (resp. e_*).

[4]In fact, we consider here the étale schemes separated and of finite type over S rather than all the étale schemes over S. However, by [SGA 4_{II}, Exposé VII, 3.1 and 3.2], they give the same category of étale sheaves. Because we are mainly interested in étale sheaves here, we do not distinguish between these two categories.

[5]The notations used here are in the same spirit as in [SGA 4_{II}, Exposé VII, §4], but are different from the ones in [SGA 4_{I}, Exposé III]. See also [Stacks, Section 0CMZ] for a comparison of notations.

[6]Warning: For $\mathcal{C} = \mathsf{Sm}$, this continuous functor does not induce a morphism of sites in general. See [Ols16, 2.2.30] or [Stacks, Section 07BF] for some examples that e^*_{Sm} is not exact.

The following Cartesian square

$$\begin{tikzcd} Y \arrow[r, "g"] \arrow[d, "h"'] & X \arrow[d, "f"] \\ T \arrow[r, "e"] & S \end{tikzcd}$$

induces a commutative diagram

$$\begin{tikzcd} \mathsf{Et}/X \arrow[r, "g"] \arrow[d, "\sigma_f"'] & \mathsf{Et}/Y \arrow[d, "\sigma_h"'] \\ \mathcal{C}/S \arrow[r, "e_{\mathcal{C}}"] & \mathcal{C}/T. \end{tikzcd}$$

By definition of the direct images, the above diagram induces the following commutative diagram

$$\begin{tikzcd} \mathsf{Shv}_{\text{ét}}(\mathcal{C}/T,\Lambda) \arrow[r, "e_*^{\mathcal{C}}"] \arrow[d, "\sigma_{h*}"'] & \mathsf{Shv}_{\text{ét}}(\mathcal{C}/S,\Lambda) \arrow[d, "\sigma_{f*}"'] \\ \mathsf{Shv}_{\text{ét}}(\mathsf{Et}/Y,\Lambda) \arrow[r, "g_*"] & \mathsf{Shv}_{\text{ét}}(\mathsf{Et}/X,\Lambda). \end{tikzcd}$$

In particular, if $f = \mathrm{id}_S$, then $\sigma_{S*}e_*^{\mathcal{C}} = e_*\sigma_{T*}$ and $\sigma_T^*e^* \simeq e_{\mathcal{C}}^*\sigma_S^*$.

Lemma A.2.1. (1) *The functors σ_{f*} and σ_f^* are exact.*

(2) *The functor σ_f^* is fully faithful.*

(3) $\sigma_{f*}\sigma_S^* \simeq f^*$.

Proof. This result is well-known. In fact, the inclusion $\sigma_f\colon (\mathsf{Et}/X)_{\text{ét}} \hookrightarrow \mathcal{C}_{\text{ét}}$ is not only continuous but also co-continuous in the sense of [SGA 4_{I}, Exposé III, Définition 2.1]. It follows from [Stacks, Lemma 04BH] and [Stacks, Lemma 077I] that σ_{f*} and σ_f^* are exact and that σ_f^* is fully faithful (σ_{f*} is the g^{-1}, and σ_f^* is the $g_!$ in loc. cit.).

Now, we prove the last assertion. Note that σ_f can be factored as

$$\mathsf{Et}/X \xrightarrow{\sigma_X} \mathcal{C}/X \xrightarrow{\iota} \mathcal{C}/S.$$

Thus $\sigma_{f*} = \sigma_{X*}\iota_*$, where $\iota_*\mathscr{F}(U/X) = \mathscr{F}(U/S)$. Since $f\colon X \to S$ in a morphism in $\mathcal{C}/S$, the functor ι_* is in fact $f_{\mathcal{C}}^*$. It follows that

$$\sigma_{f*}\sigma_S^* = \sigma_{X*}f_{\mathcal{C}}^*\sigma_S^* \simeq \sigma_{X*}\sigma_X^*f^* \simeq f^*,$$

where the last isomorphism holds by (2). □

Now, we study the derived functors. First, we derive the diagram before Lemma A.2.1. By Lemma A.2.1 (1), the functors σ_{f*} and σ_{h*} are exact.

Lemma A.2.2. *For $K \in D(\mathsf{Shv}_{\text{ét}}(\mathcal{C}/T,\Lambda))$, we have a canonical isomorphism*

$$\sigma_{f*}Re_*^{\mathcal{C}}K \simeq Rg_*\sigma_{h*}K.$$

Proof. By Lemma A.2.1, the functor σ_{h*} admits an exact left adjoint σ_h^*. Thus by some formal reason, the functor σ_{h*} preserves K-injective complexes in the sense of [Spa88]. Then this lemma follows because K-injective resolutions compute unbounded right derived functors. □

Theorem A.2.3. *Let $e\colon T \to S$ be a morphism of noetherian schemes with T excellent. Then for $\mathscr{F} \in \mathsf{Shv}_{\text{ét}}(\mathsf{Et}/T, \Lambda)$, the base change morphism*

$$\alpha_{\mathscr{F}} : \sigma_S^* Re_* \mathscr{F} \longrightarrow Re_*^{\mathcal{C}} \sigma_T^* \mathscr{F}$$

is an isomorphism in $D(\mathsf{Shv}_{\text{ét}}(\mathcal{C}/S, \Lambda))$.

Proof. Let $\Lambda(X)$ the étale sheaf associated with the presheaf mapping $U \in \mathcal{C}/S$ to the free Λ-module generated by $\mathrm{Mor}_S(U, X)$. Consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{D(\mathsf{Shv}_{\text{ét}}(\mathcal{C}/S,\Lambda))}(\Lambda(X)[n], \sigma_S^* Re_* \mathscr{F}) \arrow[r, "\alpha_1"] \arrow[d, "\beta_1", "\simeq"'] & \mathrm{Hom}_{D(\mathsf{Shv}_{\text{ét}}(\mathcal{C}/S,\Lambda))}(\Lambda(X)[n], Re_*^{\mathcal{C}} \sigma_T^* \mathscr{F}) \arrow[d, "\beta_2", "\simeq"'] \\
\mathrm{Hom}_{D(\mathsf{Shv}_{\text{ét}}(\mathsf{Et}/X,\Lambda))}(\Lambda(X)[n], \sigma_{f*} \sigma_S^* Re_* \mathscr{F}) \arrow[r, "\alpha_2"] & \mathrm{Hom}_{D(\mathsf{Shv}_{\text{ét}}(\mathsf{Et}/X,\Lambda))}(\Lambda(X)[n], \sigma_{f*} Re_*^{\mathcal{C}} \sigma_T^* \mathscr{F}).
\end{tikzcd}$$

Here, α_1 and α_2 are induced by the base change morphisms; β_1 and β_2 are adjunction isomorphisms. Consider the pullback of the base change morphism

$$\begin{aligned}
f^* Re_* \mathscr{F} &\xrightarrow{\sim} \sigma_{f*} \sigma_S^* Re_* \mathscr{F} \\
&\to \sigma_{f*} Re_*^{\mathcal{C}} \sigma_T^* \mathscr{F} \\
&\xrightarrow{\sim} Rg_* \sigma_{h_*} \sigma_T^* \mathscr{F} \\
&\xrightarrow{\sim} Rg_* h^* \mathscr{F},
\end{aligned}$$

where the first and the last arrows are isomorphisms by Lemma A.2.1 (3) and the third arrow is an isomorphism by Lemma A.2.2. According to the smooth base change theorem (Theorem A.1.2), $f^* Re_* \mathscr{F} \to Rg_* h^* \mathscr{F}$ is an isomorphism, which implies that α_2 and then α_1 are isomorphisms. Since $\{\Lambda(X)[n]\}$ is a system of generators in $D(\mathsf{Shv}_{\text{ét}}(\mathcal{C}/S, \Lambda))$, we obtain the expected isomorphism in the derived category. □

Remark A.2.4. Using a spectral sequence argument like [Stacks, Lemma 0F09], we can also establish Theorem A.1.2 and Theorem A.2.3 for bounded below complexes of sheaves of Λ-modules on $T_{\text{ét}}$.

Note that the spectrum of a field is an excellent scheme. So we have the following result:

Corollary A.2.5. *Let k be a field of exponential characteristic p, and let $\Lambda = \mathbb{Z}[\frac{1}{p}]$. Let K/k be a field extension. Write $e\colon \operatorname{Spec} K \to \operatorname{Spec} k$ the induced morphism. Let $\mathscr{F}$ be a sheaf of Λ-modules on $(\operatorname{Spec} K)_{\text{ét}}$.*

(1) *For a Cartesian diagram*

$$\begin{tikzcd}
X_K \arrow[r, "g"] \arrow[d, "h"'] & X \arrow[d, "f"] \\
\operatorname{Spec} K \arrow[r, "e"] & \operatorname{Spec} k
\end{tikzcd}$$

with f smooth and of finite type, the base change morphism

$$\alpha^i_{\mathscr{F}} : f^* R^i e_* \mathscr{F} \longrightarrow R^i g_* h^* \mathscr{F}$$

is an isomorphism for every i.

(2) *The base change morphism*

$$\alpha_{\mathscr{F}} : \sigma_k^* Re_* \mathscr{F} \longrightarrow Re_*^{\mathcal{C}} \sigma_K^* \mathscr{F}$$

is an isomorphism in $D(\mathsf{Shv}_{\text{ét}}(\mathcal{C}/k, \Lambda))$*, where* $\mathcal{C}$ *is* Sm *or* $\mathsf{Sm}_{\leq n}$ *for some* $n \in \mathbb{N}$*.*

APPENDIX B. A REPRESENTABILITY RESULT

by BRUNO KAHN

For a field k, let $\mathsf{Sm}_l(k)$ be the category of smooth separated k-schemes locally of finite type, and $\mathsf{Sm}(k)$ the full subcategory of those which are of finite type. We provide them with the étale topology. and write $\mathsf{Shv}_l(k)$ and $\mathsf{Shv}(k)$ for the corresponding categories of sheaves of sets.

Lemma B.0.1. *The restriction functor* $\mathsf{Shv}_l(k) \to \mathsf{Shv}(k)$ *is an isomorphism of categories.*

Proof. The inverse functor sends a sheaf $\mathscr{F}$ to $U \mapsto \prod_{i\in I} \mathscr{F}(U_i)$, where the U_i are the connected components of U. □

Lemma B.0.2. *Let* $\mathscr{F} \in \mathsf{Shv}(k)$*. If* $\mathscr{F} \neq \emptyset$*, then* $\mathscr{F}(\operatorname{Spec} E) \neq \emptyset$ *for some finite separable extension* E/k*.*

Proof. The assumption means that there exists $U \in \mathsf{Sm}(k)$ such that $\mathscr{F}(U) \neq \emptyset$. But U has a closed point u with separable residue field E (this follows from the characterisation of smoothness in [SGA 1, II, Def. 1.1]), hence $\mathscr{F}(\operatorname{Spec} E) = \mathscr{F}(u) \neq \emptyset$. □

Lemma B.0.3. *Let E be a finite separable extension of k, and let* $\underline{L} \in \mathsf{Shv}(k)$ *be the étale sheaf represented by* $L = \operatorname{Spec} E$*. Then there is an isomorphism of categories*

$$\mathsf{Shv}(E) \simeq \mathsf{Shv}(k)/\underline{L}.$$

This isomorphism transports a representable sheaf $\underline{F} \in \mathsf{Shv}(E)$ *to* $\underline{F} \to \underline{L} \in \mathsf{Shv}(k)/\underline{L}$*.*

Proof. Let $\mathscr{F} \xrightarrow{p} \underline{L} \in \mathsf{Shv}(k)/\underline{L}$. For $U \in \mathsf{Sm}(k)$ and $\pi \in \underline{L}(U) = \operatorname{Mor}_k(U, L)$, let $\mathscr{F}_\pi(U) = p^{-1}(\pi)$ so that $\mathscr{F}(U) = \coprod_{\pi\in \underline{L}(U)} \mathscr{F}_\pi(U)$. The isomorphism of categories is now clear: writing $U \mapsto U_{(k)}$ for the forgetful functor $\mathsf{Sm}(E) \to \mathsf{Sm}(k)$,

In one direction: Let $\mathscr{F} \xrightarrow{p} \underline{L} \in \mathsf{Shv}(k)/\underline{L}$. For $U \in \mathsf{Sm}(E)$, let $\mathscr{F}'(U) = \mathscr{F}_{\pi_U}(U_{(k)})$ where $\pi_U : U \to \operatorname{Spec} E$ is the structural morphism.

In the other direction: Let $\mathscr{F}' \in \mathsf{Shv}(E)$. For $U \in \mathsf{Sm}(k)$, let $\mathscr{F}(U) = \coprod_{\pi\in \underline{L}(U)} \mathscr{F}'(U, \pi)$, and let $p(U) : \mathscr{F}(U) \to \underline{L}(U)$ be the obvious projection.

If $F \in \mathsf{Sm}(E)$ and $U \in \mathsf{Sm}(k)$, a k-morphism $f : U \to F$ induces a unique E-structure on U through which f factors; hence the claim for representable sheaves. □

Proposition B.0.4. *Let* $\mathscr{F} \in \mathsf{Shv}(\mathsf{Sm}(k))$ *be a sheaf of groups. Suppose that there is an exact sequence*

$$1 \to \underline{G} \to \mathscr{F} \xrightarrow{p} \mathscr{L} \to 1$$

where $\underline{G}$ *is representable by a smooth connected algebraic k-group G and* $\mathscr{L}$ *is locally constant. Then* $\mathscr{F}$ *is representable by a k-group scheme in* $\mathsf{Sm}_l(k)$*, whose identity component is G.*

(The assertion makes sense thanks to Lemma B.0.1.)

Proof. It suffices to show that $\mathscr{F}$ is representable as a sheaf of sets, the group structure taking care of itself by Yoneda's lemma as well as the claim on the identity component. Since $\mathscr{L}$ is locally constant, it is representable by an étale k-scheme locally of finite type $L = \coprod_{i\in I} L_i$ where $L_i = \operatorname{Spec} E_i$ for a finite separable extension E_i/k [Mil80, VIII, p. 54, Rem. 1.12]. Then $\mathscr{F} = \coprod_{i\in I} \mathscr{F}_i$ where $\mathscr{F}_i = p^{-1}(\underline{L}_i)$ with $\underline{L}_i$ the sheaf represented by L_i; since a coproduct of representable sheaves in $\mathsf{Shv}_l(k)$ is representable, it suffices to show that each $\mathscr{F}_i$ is representable.

Recall (e.g. [SGA 7_{I}, Exposé VII, §1]) that the action of $\underline{G}$ on $\mathscr{F}$ by left translations defines a $\underline{G}$-torsor over $\mathscr{L}$, whence a $\underline{G}$-torsor structure on $\mathscr{F}_i = \mathscr{F} \times_{\mathscr{L}} \underline{L}_i$ over $\underline{L}_i$. By transport of structure, this makes the sheaf $\mathscr{F}_i' \in \mathsf{Shv}(E_i)$ associated to $\mathscr{F}_i$ by Lemma B.0.3 a $\underline{G}_E$-torsor over the point.

By Lemma B.0.2, $\mathscr{F}_i'$ is trivial over a finite separable extension of E, therefore it is representable by descent [SGA 1, VIII, Cor. 7.6] since G is quasi-projective [Cho57]. If F_i is the corresponding E_i-scheme, $\mathscr{F}_i$ is then represented by $(F_i)_{(k)}$ by applying Lemma B.0.3 again. $\square$

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